Learning the Japanese Abacus

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Contents

0 Introduction

Abacus thinking can be applied not only to calculations on the abacus, but also to calculating by hand with pencil and paper and to calculating in your head. After completing these lessons, you will have learned three ways of approaching any arithmetic calculation:

1. Calculate on the abacus.
2. Calculate with pencil and paper.
3. Calculate in your head.

With sufficient practice, you will master all three methods.

0.1 Conventions Used in this Workbook

There are examples of problems in this workbook. The problem numbers have the format *section_number.problem_number*. For example, Problem 4.3 is the third problem listed in Lesson 4.

1 Clearing the Abacus

Pick up your abacus on both sides and hold it vertically so that the four bottom beads on each rod fall away from the center bar. Now gently set your abacus flat on a table. Working from left to right, move the top bead away from the center bar to the top of each rod.

2 First Use of the Abacus

2.1 Counting from 1–20

Counting is an easy exercise that teaches you to read numbers that are set on an abacus.

On each rod, there are four 1-beads below the center bar and one 5-bead above it. Choose a rod that is marked with a dot on the *center bar*. This rod is your *units-rod*, also called the *ones-rod*, which corresponds to the ones-digit in a written number.[1] **Remember that *units-rod* and *ones-rod* mean the same thing.**

Clear your abacus by moving all the beads away from the center bar. Always move the beads on the abacus with your right hand, even if you are left-handed.[2]

With your thumb, move a 1-bead up to the center bar. This is 1.[3] With your thumb, move another 1-bead up to the center bar. This is 2. With your thumb, move another 1-bead up to the center bar. This is 3. With your thumb, move another 1-bead up to the center bar. This is 4. By this point, you have run out of 1-beads to bring up with your thumb. Remember that *the bead above the center bar counts for five 1-beads*. That is why this bead is called a 5-bead. With your right index finger, move the 5-bead to the center bar. Don't put your index finger away yet. Slide your index finger over the center bar and pull the four 1-beads away from the center bar to the bottom of the rod. This is 5. With your thumb, move a 1-bead up to the center bar. This is 6. With your thumb, move another 1-bead up to the center bar. This is 7. With your thumb, move another 1-bead up to the center bar. This is 8. With your thumb, move another 1-bead up to the center bar. This is 9. You have now fit the largest possible numbers onto the ones-rod — the number nine.

To count to the next number, 10, first clear the ones-rod and then set a 1 on the tens-rod, as will now be explained. With your right index finger, pull the four 1-beads away from the center bar in one

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[1]For visual ease, this document shows the center-bar dot of the units-rod to be darker than other dots. On your abacus, all center-bar dots appear the same.

[2]It is possible to move the beads with your left hand instead of the right, but using the right hand is more natural because numbers are read from left to right on the abacus. Working with your right hand allows you to easily look at your results while you work. Working with your left hand makes this looking harder to do.

[3]In the abacus diagrams of this document, the beads that have been most recently moved are drawn with thicker lines.

Figure 1: Rod Values of the Abacus

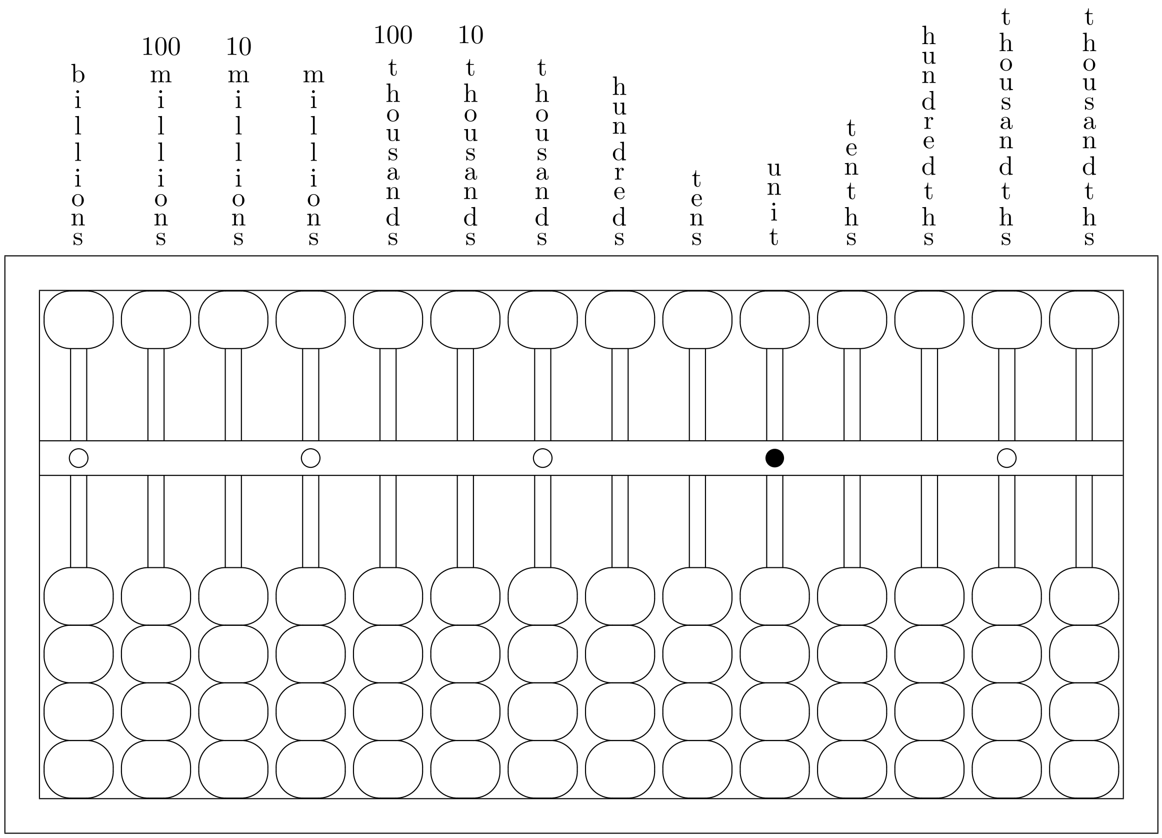
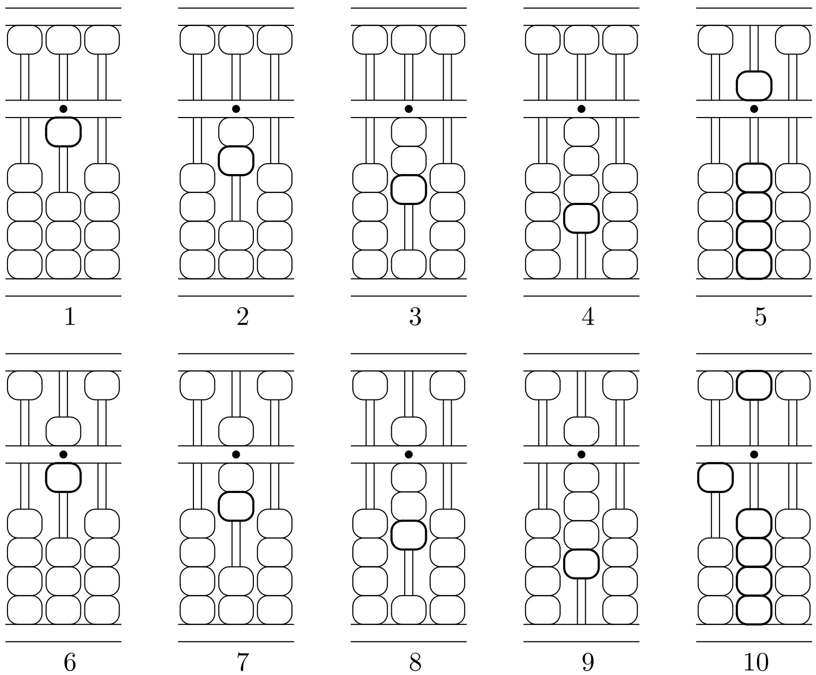
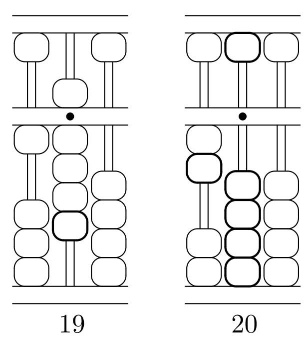


Figure 2: Counting from 1–10



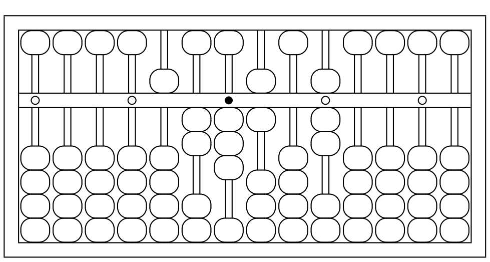
motion by pulling down the top 1-bead. Then, with your right index finger again, push the 5-bead away from the center bar also. Next, move your thumb to the tens-rod and push a 1-bead to the center bar. This is 10. Reread this paragraph as you count on your abacus from 1 to 10 and refer to Figure 2.

Now continue counting from 11 to 20. The only thing different now is that there is a 1-bead set on the tens-rod as you count from 11 to 19. Going from 19 to 20, you again clear the ones-rod with your index finger (clearing 1-beads before clearing the 5-bead) and then add a 1-bead with your thumb to the tens-rod. (See figure to the right.)



If you can count to one hundred by ones (or to ten by tenths, or to one by hundredths) without hesitation, you are proficient in reading numbers on the abacus.

Here are examples of various numbers set on the abacus. On your abacus, all the dots are the same color, but in these examples, the dot on the units-rod has been darkened for easy visibility.



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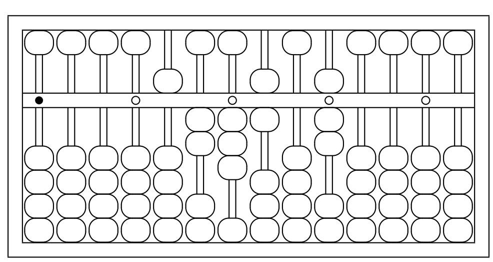

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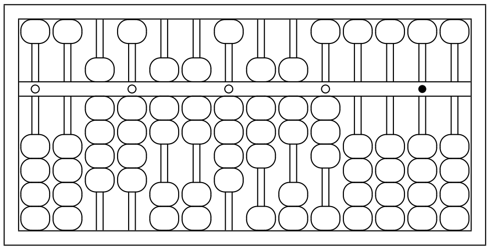
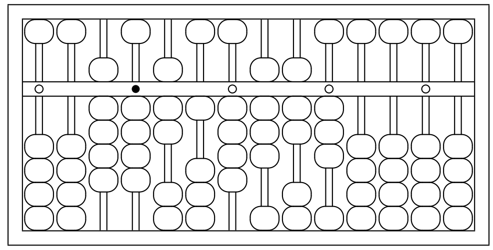


94, 714, 873, 000

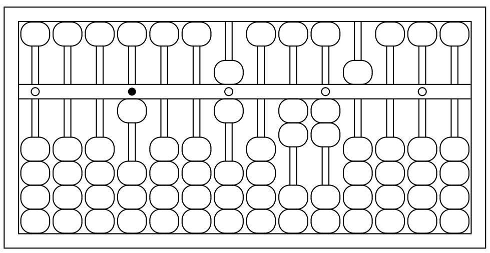


94.714873

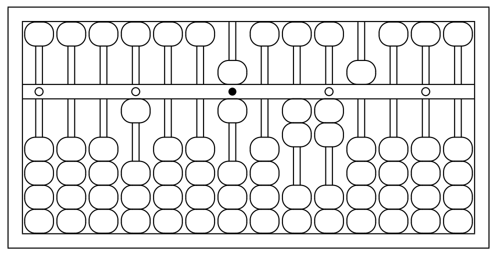
2.2 Exercises

Exercise 2.1

Write down the number set on the abacuses below. The units-rod is indicated with a dark dot.



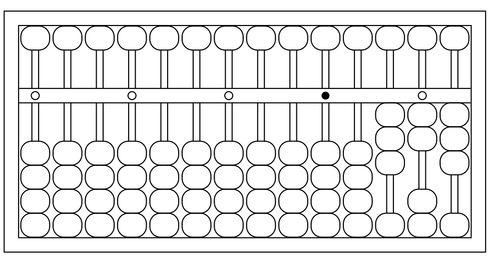
1.0060225



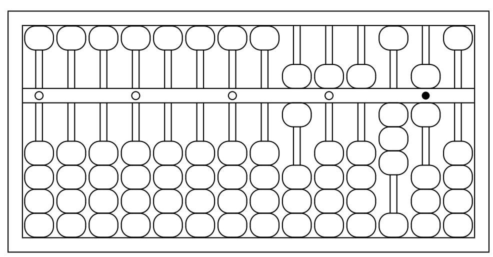
1, 006.0225

Exercise 2.2

Set the following numbers on your abacus: 0.0323 and 65, 536. *Note: Pick any units rod that works.*



0.0323



65, 536

3 Rate of Composing a Higher-Value Unit

3.1 The Abacus

Clear the abacus and choose a ones-rod. How many 1-beads must you count on the ones-rod before you get a 1-bead on the tens-rod? (A 1-bead on the tens-rod is a *higher-value unit* and a 1-bead on the ones-rod is a *lower-value unit*.) You need to count ten 1-beads on the ones-rod to get one 1-bead on the tens-rod. We say that the *rate of composing* a ten-bead from 1-beads is ten.

What is the rate of composing a 1-bead on the *hundreds*-rod from 1-beads on the ones-rod? (A 1-bead on the hundreds-rod is a higher-value unit than a 1-bead on the ones-rod.) Keep counting with 1-beads from the ones-rod until you get a 1-bead on the hundreds rod. Write your answer in the box below.

$\boxed{100}$ (Answer)

If you know how to multiply, there is something more to discover from the rate of composing a higher-value unit. What is the rate of composing a 1-bead on the hundreds-rod from 1-beads on the tens-rod? A 1-bead on the hundreds-rod (call this a hundred-bead) is a higher-value unit than a 1-bead on the tens-rod (call this a ten-bead). Keep counting with ten-beads until you get a hundred-bead. Write your answer in the box below.

$\boxed{10}$ (Answer)

The rate of composing a hundred-bead from ten-beads is 10. The rate of composing a ten-bead from one-beads is also 10. To find the rate of composing a hundred-bead from one-beads, you can do it the hard way or the easy way.

The hard way is to count one-beads until you get one hundred-bead. The easy way is to find the rate of composing a ten-bead from one-beads, which is 10, and multiply it by the rate of composing a hundred-bead from ten-beads, which also happens to be 10. $10 \times 10 = 100$, so rate of composing a hundred-bead from one-beads is 100.
What is the rate of composing a thousand-bead from ten-beads?

$\boxed{100}$ (Answer)

What is the rate of composing a ten-thousand-bead from ten-beads?

$\boxed{1,000}$ (Answer)

What is the rate of composing a million-bead from ten-beads?

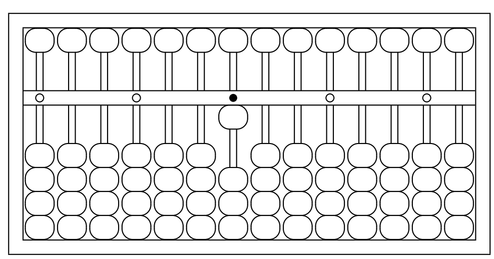
$\boxed{100,000}$ (Answer)

3.2 Keeping Time

The rate of composing a higher-value unit is also useful for thinking about how we keep time. We record time in the format HH:MM. For example, 3:16 is sixteen minutes past three o'clock; 11:55 is 55 minutes past eleven o'clock. Of course, there are sixty minutes in one hour.

Choose a dot on your abacus. Let this be the hour-rod. The rod to the right of the hour-rod is the ten-minute-rod. The rod to the right of the ten-minute-rod is the minute-rod. The rod to the *left* of the hour-rod is the ten-hour-rod. Set the time 1:00 on the abacus. (See figure to the right.) By counting 1-beads on the minute-rod, what is the rate of composing a 1-bead on the ten-minute rod?

$\boxed{10}$ (Answer)



1 : 00

Set the time again to 1:00. Counting 1-beads on the ten-minute-rod, what is the rate of composing a 1-bead on the hour-rod? *Be careful!*

$\boxed{6}$ (Answer)

What is the rate of composing a 1-bead on the hour-rod from 1-beads on minute-rod?

$$\boxed{10 \times 6 = 60} \qquad \text{(Answer)}$$

3.3 Exercises

Exercise 3.1

On the abacus, what is the rate of composing a 1-bead from tenth-beads?

$$\boxed{10} \qquad \text{(Answer)}$$

Exercise 3.2

On the abacus, what is the rate of composing a 1-bead from hundredth-beads?

$$\boxed{100} \qquad \text{(Answer)}$$

Exercise 3.3

On the abacus, what is the rate of composing a 10,000-bead from hundredth-beads?

$$\boxed{1,000,000} \qquad \text{(Answer)}$$

Exercise 3.4

Think of a digital clock. What is the rate of composing 10 hours from minutes?

$$\boxed{10 \times 6 \times 10 = 600} \qquad \text{(Answer)}$$

Exercise 3.5

What is the rate of composing an hour from seconds?

$$\boxed{60 \times 60 = 3600} \qquad \text{(Answer)}$$

Exercise 3.6

There are 12 inches in a foot, there are 3 feet in a yard, and there are 100 yards between end zones on a football field. What is the rate of composing a football field from inches?

$$\boxed{12 \times 3 \times 100 = 3600} \qquad \text{(Answer)}$$

4 Facts for Five and Ten

The facts for five and ten are the core of calculating on the abacus. These facts are based on a *principal number*, either five of ten, a number between zero and the principal number, inclusive[4], and the complement of that number with regard to the principal number. The facts are made up of groups of six statements.

[4] *Inclusive* means to include the numbers at the beginning and the end with the numbers that lie between them. For example, the counting numbers from 18 to 22 *inclusive* means the set of numbers 18, 19, 20, 21, and 22. On the other hand, the counting numbers from 18 to 22 *exclusive* means the set of numbers 19, 20, and 21.

4.1 Facts for Five

Think of the number five. This is your principal number. Now choose a number between zero and five, inclusive. Suppose you thought of the number three. The complement is that number that you add to three to get five. The fives-complement of three is two. Say the following out loud:

1. "Three plus two is five." ($3 + 2 = 5$)
2. "Two plus three is five." ($2 + 3 = 5$)
3. "Three is five minus two." ($3 = 5 - 2$)
4. "Minus three is two minus five." ($-3 = 2 - 5$)
5. "Two is five minus three." ($2 = 5 - 3$)
6. "Minus two is three minus five." ($-2 = 3 - 5$)

Statements 3–6 must be stated exactly in the way prescribed above because they follow the thinking and finger-motion used while adding or subtracting on the abacus. It is true to say, for example, "Two minus five is minus three." or "Minus three is minus five plus two.", but neither helps you master calculating on the abacus.

4.2 Facts for Ten

Think of the number ten. This is your principal number. Now choose of a number between zero and ten, inclusive. Suppose you thought of the number four. The tens-complement of four is six. Say the following out loud:

1. "Four plus six is ten." ($4 + 6 = 10$)
2. "Six plus four is ten." ($6 + 4 = 10$)
3. "Four is minus six plus ten." ($4 = -6 + 10$)
4. "Minus four is minus ten plus six." ($-4 = -10 + 6$)
5. "Six is minus four plus ten." ($6 = -4 + 10$)
6. "Minus six is minus ten plus four." ($-6 = -10 + 4$)

For the same reason described previously,[5] Statements 3–6 must be stated exactly in the way prescribed above.

4.3 Rule for Remembering the Facts

Note that for a principal number of five, Statements 3–6 always state a positive number right after saying "is". For a principle number of ten, this flips around — Statements 3–6 always state a negative number after saying "is". This difference is emphasized by the rule:

For 5, postive first; for 10, negative first.

[5]Lesson 4.1 on p. 8

4.4 The Five and Ten Memory Game

The traditional Memory Game is a game of finding pairs of cards. The Five and Ten Memory Game is the Memory Game where the pairs of cards are *not* usually identical, but they are related to each other as complements of five or as complements of ten.

Listed below are the possible pairs of cards in the Five and Ten Memory Game. It is easy to make the game yourself with cardstock and a pen.[6] The principal number is in brackets at the top of each card; it is always either five or ten. A matching pair has the same principal number at the top and a pair of complements of the principal number at the bottom. The six facts for each pair are displayed below next to each pair. The order in which they are said is not important. What is important is that each of the facts is said correctly *word-for-word*![7]

[5] **0**	**[5]** **5**	Zero plus five is five. Five plus zero is five. Zero is five minus five. Minus zero is five minus five. Five is five minus zero. Minus five is zero minus five.
[5] **1**	**[5]** **4**	One plus four is five. Four plus one is five. One is five minus four. Minus one is four minus five. Four is five minus one. Minus four is one minus five.
[5] **2**	**[5]** **3**	Two plus three is five. Three plus two is five. Two is five minus three. Minus two is three minus five. Three is five minus two. Minus three is two minus five.
[10] **0**	**[10]** **10**	Zero plus ten is ten. Ten plus zero is ten. Zero is minus ten plus ten. Minus zero is minus ten plus ten. Ten is minus zero plus ten. Minus ten is minus ten plus zero.
[10] **1**	**[10]** **9**	One plus nine is ten. Nine plus one is ten. One is minus nine plus ten. Minus one is minus ten plus nine. Nine is minus one plus ten. Minus nine is minus ten plus one.
[10] **2**	**[10]** **8**	Two plus eight is ten. Eight plus two is ten. Two is minus eight plus ten. Minus two is minus ten plus eight. Eight is minus two plus ten. Minus eight is minus ten plus two.

[6] This game is also available for purchase from Peeter Pirn.

[7] See Lesson 4.1 on p. 8 for the reason why the facts must be stated word-for-word.

[**10**] **3**	[**10**] **7**	Three plus seven is ten. Seven plus three is ten. Three is minus seven plus ten. Minus three is minus ten plus seven. Seven is minus three plus ten. Minus seven is minus ten plus three.
[**10**] **4**	[**10**] **6**	Four plus six is ten. Six plus four is ten. Four is minus six plus ten. Minus four is minus ten plus six. Six is minus four plus ten. Minus six is minus ten plus four.
[**10**] **5**	[**10**] **5**	Five plus five is ten. Five plus five is ten. Five is minus five plus ten. Minus five is minus ten plus five. Five is minus five plus ten. Minus five is minus ten plus five.

4.5 Exercises

Exercise 4.1

What is the fives-complement of 2? State the facts for this pair of fives-complements.

3 is the fives-complement of 2.

Two plus three is five.
Three plus two is five.
Two is five minus three.
Minus two is three minus five.
Three is five minus two.
Minus three is two minus five.

Exercise 4.2

What is the tens-complement of 0? State the facts for this pair of tens-complements.

10 is the tens-complement of 0.

Zero plus ten is ten.
Ten plus zero is ten.
Zero is minus ten plus ten.
Minus zero is minus ten plus ten.
Ten is minus zero plus ten.
Minus ten is minus ten plus zero.

Exercise 4.3

Make or obtain the Five and Ten Memory Game. Find someone who will play it with you, and play regularly until you have mastered the facts.

5 Rules for Finger Motion

The abacus is normally operated with the right hand. Using the left hand is possible, but this counteracts the left-to-right flow of calculations and obstructs the view of the beads.

There are seven rules that govern the order in which you move beads while calculating on the abacus.[8] With practice, they become second-nature. It is important that you uniformly follow these rules from the beginning; nonuniformity increases the risk of errors and causes difficulty with mental calculations.[9]

1. Use your thumb *only* to move 1-beads up. Use your index finger for all other bead motion.[10]
2. Move down a 5-bead and move up one or more 1-beads at the same time — think of squeezing the beads toward the center bar.[11]
3. First move down one or more 1-beads, and then, if necessary, move up a 5-bead.[12]
4. In quick succession, first move down a 5-bead, and then move down one or more 1-beads.[13]
5. In quick succession, first move up one or more 1-beads (using your thumb), and then move up a 5-bead (using your index finger).[14]
6. In addition, after finishing work on the units-rod, move up a 1-bead on the tens-rod.[15,16]
7. In subtraction, after subtracting a 1-bead from the tens-rod, operate on the units-rod.[17,18]

6 Addition and Subtraction

6.1 Learning the Skills

There are four skills necessary for adding and subtracting on the abacus.

1. mastery of the facts for five and ten
2. knowing which facts to apply to the task at hand
3. using the stack
4. handling and decoding negative numbers

The first skill was taught in Chapter 4. The second skill requires mastery of the first; it is taught in Lesson 6.1.1. The third skill requires mastery of the first two; it is taught in Lesson 6.1.4. The fourth skill requires mastery of the first three; it is taught in Lesson 6.3.

Knowing the facts of five and ten and knowing how to properly apply them are the skills used most often. As you practice, these skills will the the first ones to become second-nature. You will need to use the stack to make a change across several rods while adding or subtracting (as happens when adding 1 to $9,999$ or when subtracting 1 from $10,000$). You can handle most additions and subtractions without having to go into negative numbers, but when you do, the abacus can handle them, unlike the standard paper-and-pencil method. With enough practice, all four skills become second-nature.

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[8] Takashi Kojima, *The Japanese Abacus — Its Use and Theory*, (Rutland, VT and Tokyo: Charles E. Tuttle Company, 1954), 29. Rule 1 has been prepended to Kojima's list. Other parts of Kojima's book are also used in this workbook. His fine book is short and to the point; unfortunately, it is out of print. Check if your local library has a copy.

[9] See Lesson 10.

[10] See Problems 6.1, 6.3, and 6.4.

[11] See Problems 6.5.

[12] See Problems 6.6 and 6.10.

[13] See Problems 6.2.

[14] See Problems 6.4.

[15] This rule and the next one are the motivation for the Rule for Remembering Facts in Lesson 4.3 on p. 8.

[16] See Problems 6.11.

[17] This rule and the previous one are the motivation for the Rule for Remembering Facts in Lesson 4.3 on p. 8.

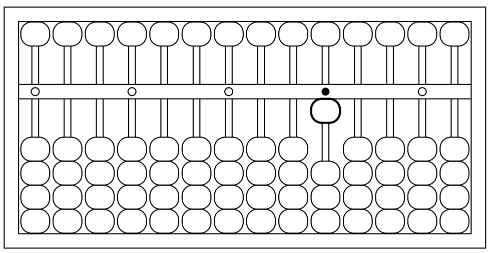
[18] See Problems 6.12.

6.1.1 Knowing Which Facts to Apply

As you learn how to apply the facts of five and ten, refer often to the rules of finger motion listed in Chapter 5 and *follow them*! There is a direct correspondence between which fact you apply and how you move your fingers. Following the rules of finger motion gives you a consistent physical "feel" for applying the facts.

$$\boxed{\begin{array}{lll} 1+1 & 2+1 & 3+1 \\ 1+2 & 2+2 & \\ 1+3 & & \end{array}} \tag{6.1}$$

Consider $1+1$. Set 1 on the units-rod. (You can choose the units-rod to be any rod marked with a dot.) You want to add 1 to it. Is there a 1-bead available for addtion? Yes. The fact to use here is trivial: $1 = 1$. Move a 1-bead up with your thumb.[19] The answer is 2.



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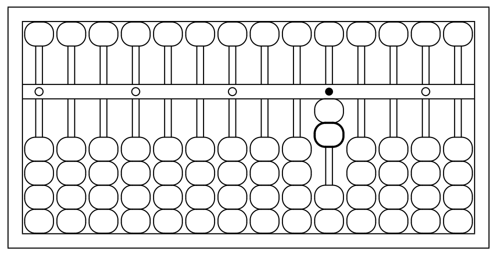
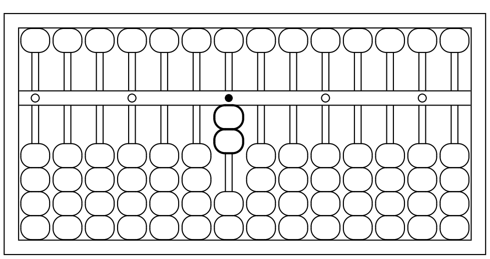


$+1 = 2$

Each computation in 6.1 is accomplished using the same finger motion. Do them all.

$$\boxed{\begin{array}{rrrr} 4+1 & 4+2 & 4+3 & 4+4 \\ & 3+2 & 3+3 & 3+4 \\ & & 2+3 & 2+4 \\ & & & 1+4 \end{array}} \tag{6.2}$$

Consider $2+3$. Set 2 on the units-rod.[20] You want to add 3 to it. Are there three 1-beads available for addtion? No. Is there a 5-bead available for addition? Yes. The fact to use is: $3 = 5 - 2$. Move down a 5-bead with your index finger; then move the finger over the center bar and move down two 1-beads.[21] The answer is 5.



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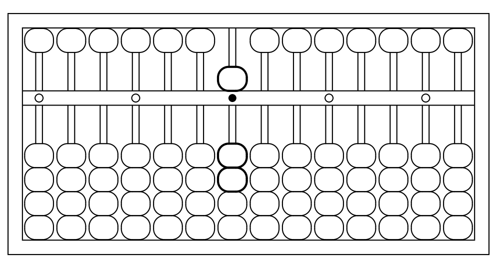


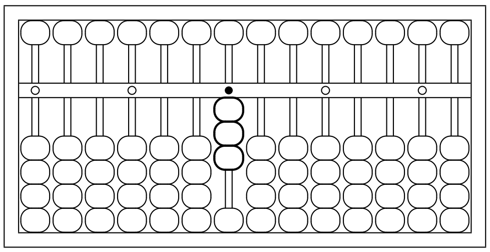
$+3 = 5$

Each computation in 6.2 is accomplished using the same finger motion. Do them all.

[19] Only the first rule of finger motion applies here. Refer to Lesson 5.
[20] The first rule of finger motion applies here. Refer to Lesson 5.
[21] The fourth rule of finger motion applies here. Refer to Lesson 5.

$$\boxed{\begin{array}{cccc} 1-1 & 2-1 & 3-1 & 4-1 \\ & 2-2 & 3-2 & 4-2 \\ & & 3-3 & 4-3 \\ & & & 4-4 \end{array}} \tag{6.3}$$

Consider $3-1$. Set 3 on the units-rod.[22] You want to subtract 1 from it. Is there a 1-bead available for subtraction? Yes. The fact to use here is trivial: $1 = 1$. Move a 1-bead down with your index finger.[23] The answer is 2.



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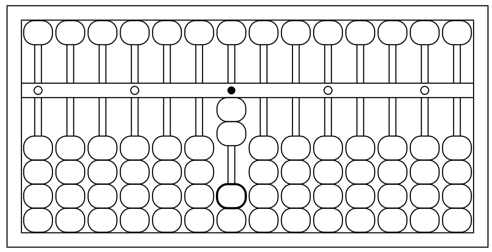
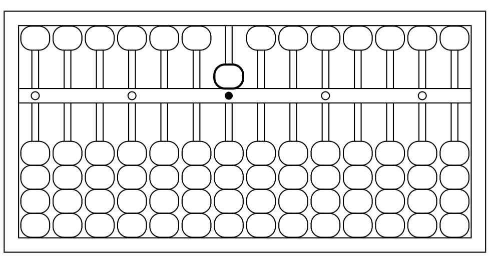


$-1 = 2$

Each computation in 6.3 is accomplished using the same finger motion. Do them all.

$$\boxed{\begin{array}{cccc} 5-1 & 5-2 & 5-3 & 5-4 \\ & 6-2 & 6-3 & 6-4 \\ & & 7-3 & 7-4 \\ & & & 8-4 \end{array}} \tag{6.4}$$

Consider $5-4$. Set 5 on the units-rod using your index finger.[24] You want to subtract 4 from it. Are there four 1-beads available for subtraction? No. The fact to use is: $-4 = 1 - 5$. In rapid succession, move a 1-bead up with your thumb and move the 5-bead up with your index finger.[25] The answer is 1.



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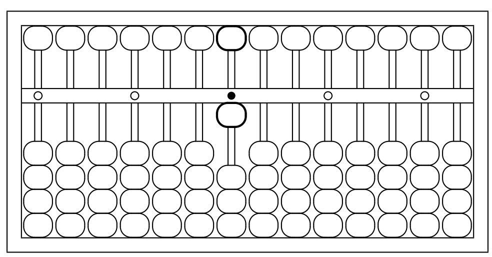


$-4 = 1$

Each computation in 6.4 is accomplished using the same finger motion. Do them all.

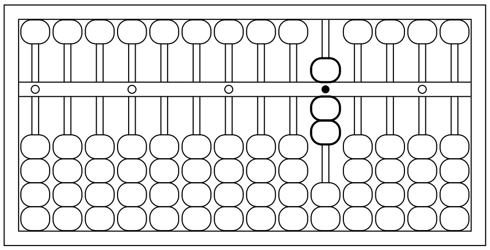
$$\boxed{\begin{array}{cccc} 5+1 & 6+1 & 7+1 & 8+1 \\ 5+2 & 6+2 & 7+2 & \\ 5+3 & 6+3 & & \\ 5+4 & & & \end{array}} \tag{6.5}$$

These problems are closely related to Problems 6.1. The main difference is how the initial number is set — use the second rule of finger motion for these problems when a 5-bead and one or more 1-beads

[22] The first rule of finger motion applies here. Refer to Lesson 5.
[23] The third rule of finger motion applies here. Refer to Lesson 5.
[24] The first rule of finger motion applies here. Refer to Lesson 5.
[25] The fifth rule of finger motion applies here. Refer to Lesson 5.

need to be set at the same time. To set seven, squeeze the 5-bead and two 1-beads at the same time toward the middle bar.

Consider $7+2$. Set 7 on the units-rod.[26] You want to add 2 to it. Are there two 1-beads available for addtion? Yes. The fact to use here is trivial: $2=2$. Move two 1-beads up with your thumb.[27] The answer is 9.



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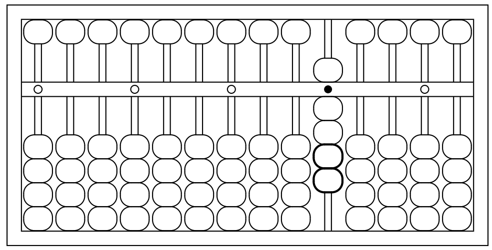


$+2=9$

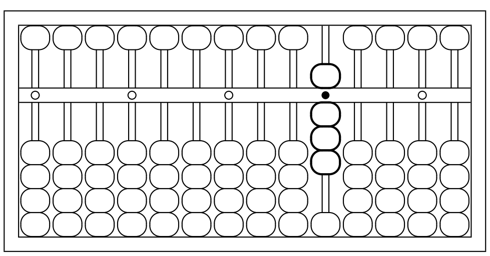
Each computation in 6.5 is accomplished using the same finger motion. Do them all.

$6-1$	$7-1$	$8-1$	$9-1$
	$7-2$	$8-2$	$9-2$
		$8-3$	$9-3$
			$9-4$

(6.6)

These problems are closely related to Problems 6.3. The main difference is how the initial number is set — use the second rule of finger motion for these problems when a 5-bead and one or more 1-beads need to be set at the same time. To set eight, squeeze the 5-bead and three 1-beads at the same time toward the middle bar.

Consider $8-2$. Set 8 on the units-rod.[28] You want to subtract 2 from it. Are there two 1-beads available for subtraction? Yes. The fact to use here is trivial: $2=2$. Move two 1-beads down with your index finger.[29] The answer is 6.



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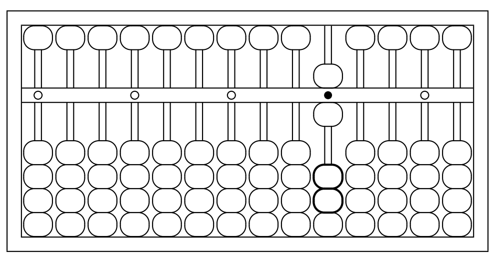


$-2=6$

Each computation in 6.6 is accomplished using the same finger motion. Do them all.

$1+5$	$2+5$	$3+5$	$4+5$

(6.7)

The 6.7 calculations are simple. Just set the first number with your thumb and set the five with your index finger.[30]

$5-5$	$6-5$	$7-5$	$8-5$	$9-5$

(6.8)

[26] The second rule of finger motion applies here. Refer to Lesson 5.
[27] The first rule of finger motion applies here. Refer to Lesson 5.
[28] The second rule of finger motion applies here. Refer to Lesson 5.
[29] The third rule of finger motion applies here. Refer to Lesson 5.
[30] The first rule of finger motion applies here. Refer to Lesson 5.

The 6.8 calculations are also simple. Just set the first number by squeezing with your thumb and index finger, unless the first number is five, of course.[31] Subtract 5 by pushing a 5-bead up with your index finger.[32]

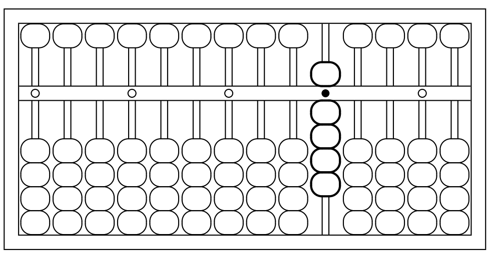
$$\boxed{\begin{array}{lll} 1+6 & 2+6 & 3+6 \\ 1+7 & 2+7 & \\ 1+8 & & \end{array}} \tag{6.9}$$

The 6.9 calculations are also simple. Just set the first number by pushing 1-beads up with your thumb.[33] Add the second number by squeezing with your thumb and forefinger.[34]

$$\boxed{\begin{array}{rrrr} 9-9 & 9-8 & 9-7 & 9-6 \\ & 8-8 & 8-7 & 8-6 \\ & & 7-7 & 7-6 \\ & & & 6-6 \end{array}} \tag{6.10}$$

These problems are closely related to Problems 6.6. The difference is that there is an additional removal of a 5-bead at the end.

Consider $9-6$. Set 9 on the units-rod.[35] You want to subtract 6 from it. Are there a 5-bead and a 1-bead available for subtraction? Yes. The fact to use here is trivial: $6=6$. Move the 1-bead down with your index finger. Then move the 5-bead up with your index finger.[36] The answer is 3.



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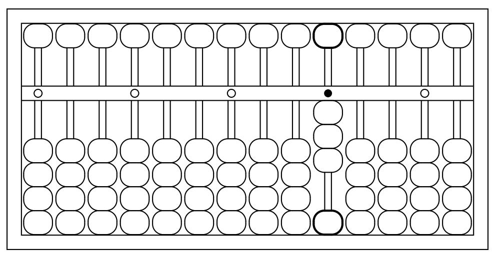


$-6=3$

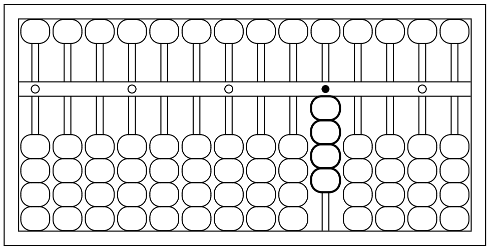
Each computation in 6.10 is accomplished using the same finger motion.

$$\boxed{1+9 \quad 2+8 \quad 3+7 \quad 4+6} \tag{6.11}$$

The previous calculations only utilized beads on the ones-rod. The only facts necessary were either trivial or facts with a principal number of 5. Now you will need to begin using facts with a principal number of 10. You will move beads on both the ones- and tens-rods.

Consider $4+6$. Set 4 on the units-rod. You want to add 6. Is there a 5-bead *and* a 1-bead available to be pushed to the center bar of the ones-rod? No. Choose the fact that fits the circumstance. The fact to use is $6=-4+10$. Subtract the 4 with your index finger and then move your thumb to the tens-rod (one rod to the left) and add 1.[37] The answer is 10.

[31] The second rule of finger motion applies here. Refer to Lesson 5.
[32] The first rule of finger motion applies here. Refer to Lesson 5.
[33] The first rule of finger motion applies here. Refer to Lesson 5.
[34] The second rule of finger motion applies here. Refer to Lesson 5.
[35] The second rule of finger motion applies here. Refer to Lesson 5.
[36] The third rule of finger motion applies here. Refer to Lesson 5.
[37] The sixth rule of finger motion applies here. Refer to Lesson 5.



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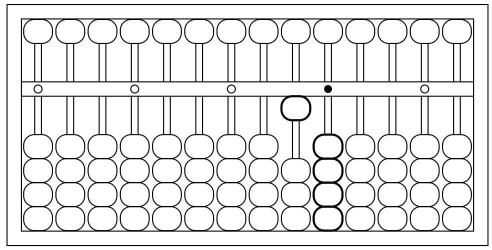
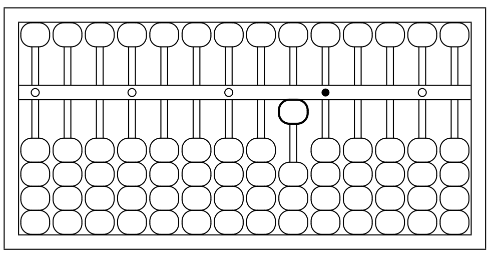


$+6 = 10$

Each computation in 6.11 is accomplished using the same finger motion.

$$\boxed{10-9 \quad 10-8 \quad 10-7 \quad 10-6} \tag{6.12}$$

Consider $10-7$. Set 1 on the tens-rod. This is 10. You want to subtract 7. Is there a 5-bead *and* two 1-beads available to be removed from the center bar of the ones-rod? No. Choose the fact that fits the circumstance. The fact to use is $-7 = -10+3$. Subtract the 1 from the tens-rod with your index finger and then move your thumb to the ones-rod and add 3.[38] The answer is 3.



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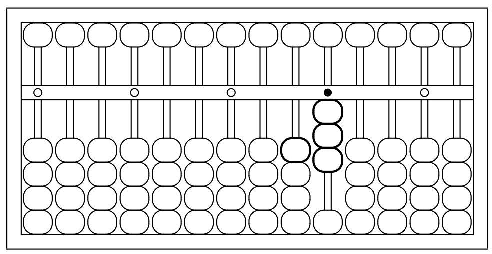
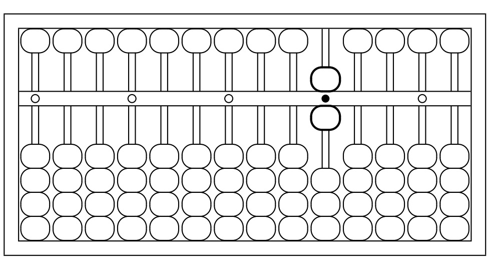


$-7 = 3$

Each computation in 6.12 is accomplished using the same finger motion.

$$\boxed{\begin{array}{cccc} 9+1 & 9+2 & 9+3 & 9+4 \\ & 8+2 & 8+3 & 8+4 \\ & & 7+3 & 7+4 \\ & & & 6+4 \end{array}} \tag{6.13}$$

Consider $6+4$. Set 6 on the ones-rod. You want to add 4. Are there four one-beads available for addition on the ones-rod? No. Is there a five-bead available for addition on the ones-rod? No. (If there was a five-bead there, you could use the fact $4 = 5-1$.) Choose the fact that fits the circumstance. The fact to use is $4 = -6+10$. First, subtract the 6 from the ones-rod with your index finger, removing the one-bead first and the five-bead second.[39] Then move your thumb to the tens-rod and add 1.[40] The answer is 10.



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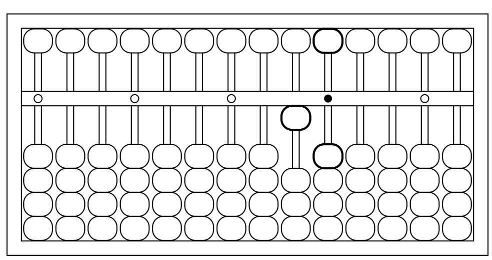


$+4 = 10$

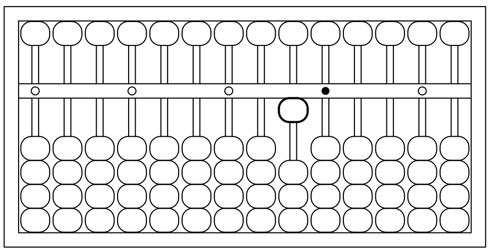
[38]The seventh rule of finger motion applies here. Refer to Lesson 5.
[39]The third rule of finger motion applies here. Refer to Lesson 5.
[40]The sixth rule of finger motion applies here. Refer to Lesson 5.

Each computation in 6.13 is accomplished using the same finger motion.

$10-4$	$10-3$	$10-2$	$10-1$
$11-4$	$11-3$	$11-2$	
$12-4$	$12-3$		
$13-4$			

(6.14)

Consider $10-2$. Set 1 on the tens-rod. This is 10. You want to subtract 2. Are there two 1-beads available to be removed from the center bar of the ones-rod? No. Choose the fact that fits the circumstance. The fact to use is $-2=-10+8$. First, subtract the 1 from the tens-rod with your index finger and then squeeze together three one-beads and a 5-bead on the ones-rod to add 8.[41] The answer is 8.



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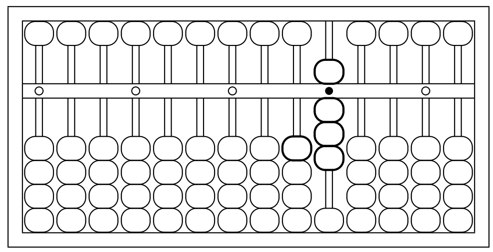


$-2=8$

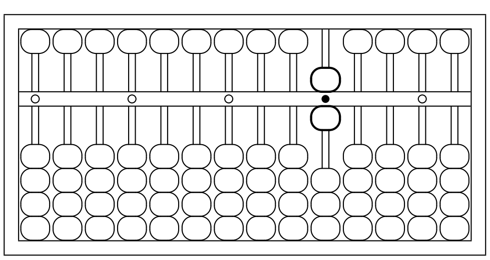
Each computation in 6.14 is accomplished using the same finger motion.

$5+6$	$6+6$	$7+6$	$8+6$
$5+7$	$6+7$	$7+7$	
$5+8$	$6+8$		
$5+9$			

(6.15)

These calculations require a combination of facts; one fact for principal number 5 and another fact for principal number 10. Consider $6+7$. Set 6 on the ones-rod. You want to add 7. Are there a 5-bead and two 1-beads available to be squeezed to the center bar of the ones-rod? No. Choose the fact that fits the circumstance. The fact to use is $7=-3+10$. Just like the fact indicates, -3 must be done first and $+10$ must be done second.

You want to subtract 3 from the ones rod. Are there three 1-beads available to be moved away from the center bar? No. Is there a 5-bead available to be moved away from the center bar? Yes. Therefore, the fact to use is $-3=2-5$. On the ones-rod, in quick succession, move two 1-beads up with your thumb and then move the 5-bead up with your index finger. On the ten's rod, move up a 1-bead with your thumb.[42] The answer is 13.



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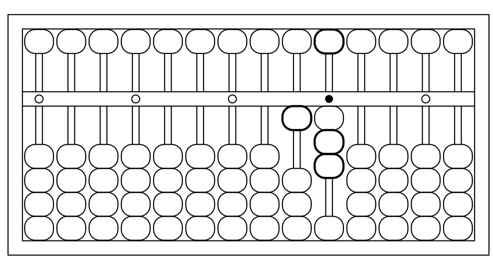


$+7=13$

[41]The seventh rule of finger motion applies here. Refer to Lesson 5.

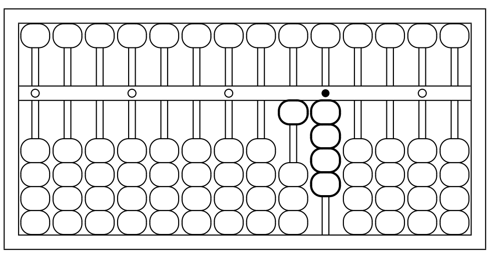
[42]The fifth and sixth rules of finger motion have been applied here. Refer to Lesson 5.

Each computation in 6.15 is accomplished using the same finger motion.

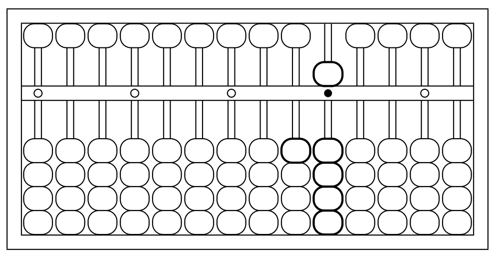
$$\boxed{\begin{array}{llll} 11-6 & & & \\ 12-6 & 12-7 & & \\ 13-6 & 13-7 & 13-8 & \\ 14-6 & 14-7 & 14-8 & 14-9 \end{array}} \tag{6.16}$$

These calculations also require a combination of facts; one fact for principal number 5 and another fact for principal number 10. Consider $14-9$. Set 14 on the tens- and ones-rods. You want to subtract 9. Are there a 5-bead and four 1-beads available to be pushed away from the center bar of the ones-rod? No. Choose the fact that fits the circumstance. The fact to use is $-9 = -10 + 1$. Just like the fact indicates, -10 must be done first and $+1$ must be done second.

You want to subtract 1 from the tens-rod. On the tens-rod, is there a 1-bead available to be moved away from the center bar? Yes. Do it right away. Now you want to add 1 to the ones rod. On the ones-rod, is there a 1-bead available to be pushed to the center bar? No. On the ones-rod, is there a 5-bead available to be pushed to the center bar? Yes. Therefore, the fact to use is $1 = 5 - 4$. On the ones-rod, in quick succession, move the 5-bead down with your index finger, slide the finger over the center bar, and then move down four 1-beads.[43] The answer is 5.



14



$-9 = 5$

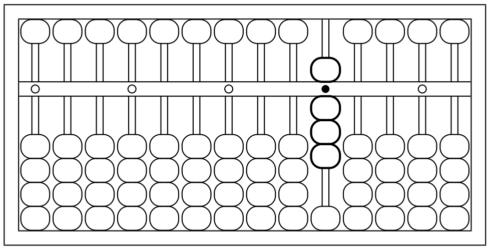
Each computation in 6.16 is accomplished using the same finger motion.

$$\boxed{\begin{array}{lllllll} 2+9 & 3+9 & 4+9 & 6+9 & 7+9 & 8+9 & 9+9 \\ & 3+8 & 4+8 & & 7+8 & 8+8 & 9+8 \\ & & 4+7 & & & 8+7 & 9+7 \\ & & & & & & 9+6 \end{array}} \tag{6.17}$$

Consider $8 + 8$. Set 8 on the ones-rod. You want to add 8. Are there a 5-bead and three 1-beads available to be squeezed to the center bar of the ones-rod? No. Choose the fact that fits the circumstance. The fact to use is $8 = -2 + 10$. Just like the fact indicates, -2 must be done first and $+10$ must be done second.

You want to subtract 2 from the ones-rod. On the ones-rod, are there two 1-beads available to be moved away from the center bar? Yes. Do it right away. Now you want to add 1 to the tens-rod. On the tens-rod, is there a 1-bead available to be pushed to the center bar? Yes. Push that 1-bead to the center bar of the tens-rod.[44] The answer is 16.

[43]The seventh and fourth rules of finger motion have been applied here. Refer to Lesson 5.
[44]The sixth rule of finger motion has been applied here. Refer to Lesson 5.



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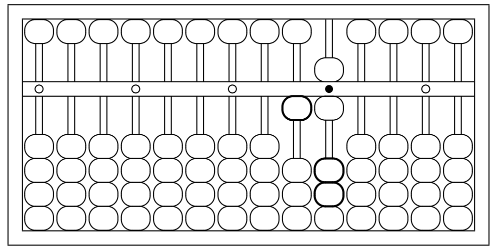


$+8 = 16$

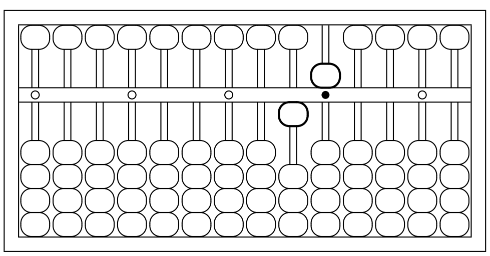
Each computation in 6.17 is accomplished using the same finger motion.

$15-6$	$15-7$	$15-8$	$15-9$
	$16-7$	$16-8$	$16-9$
		$17-8$	$17-9$
			$18-9$

[45] (6.18)

Consider $15-6$. Set 15 on the tens- and ones-rods. You want to subtract 6. Are there a 5-bead and a 1-bead available to be taken away from the center bar of the ones-rod? No. Choose the fact that fits the circumstance. The fact to use is $-6 = -10 + 4$. Just like the fact indicates, -10 must be done first and $+4$ must be done second.

You want to subtract 1 from the tens-rod. On the tens-rod, is there a 1-bead available to be moved away from the center bar? Yes. Do it right away. Now you want to add 4 to the ones-rod. On the ones-rod, are there four 1-beads available to be pushed to the center bar? Yes. Push these 1-beads to the center bar of the ones-rod.[46] The answer is 9.



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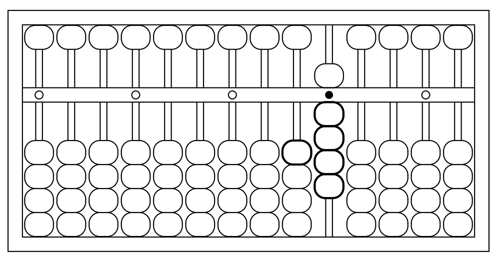


$-6 = 9$

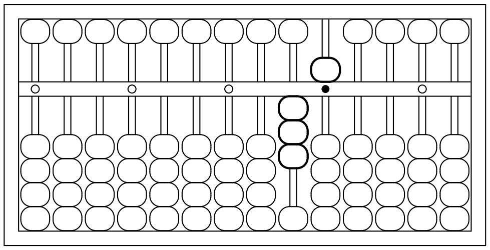
Each computation in 6.18 is accomplished using the same finger motion.

6.1.2 Adding and Subtracting Two-Digit Numbers

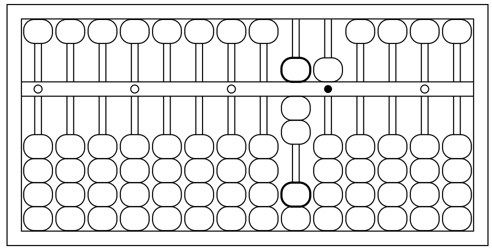
Let us now begin in earnest adding and subtracting two-digit numbers. When adding or subtracting numbers on the abacus, *start on the highest-valued rod and work to the lowest-valued rod.*

$$\boxed{35 + 42 = 77} \tag{6.19}$$

[45]CAUTION: The video lesson skips these examples by mistake. Do not skip these problems — do them all.
[46]The seventh rule of finger motion has been applied here. Refer to Lesson 5.



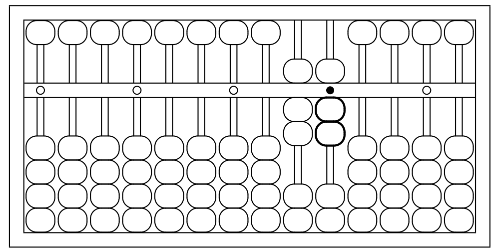
35



+40

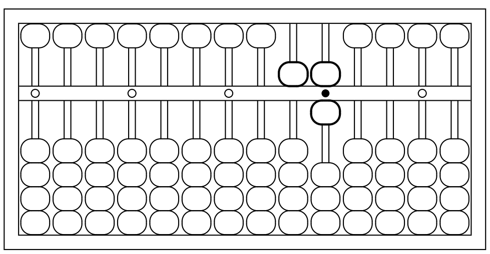
For this calculation, set 35 first. Since the tens-rod is the highest-valued rod of 42, add 4 to the tens-rod next. To do this, the fact you need to use is $4 = 5 - 1$. Execute $4 = 5 - 1$ while observing the fourth rule of finger motion. This gives an intermediate result of 75.

Now add 2 to the ones-rod. In this case, there are two 1-beads available on the ones-rod to be pushed to the center bar. Move these up with your thumb and you are finished. The answer is 77.



$+2 = 77$

$$\boxed{56 + 38 = 94} \qquad (6.20)$$



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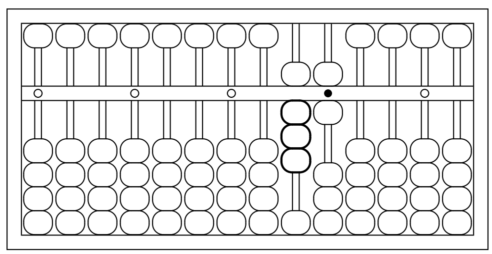
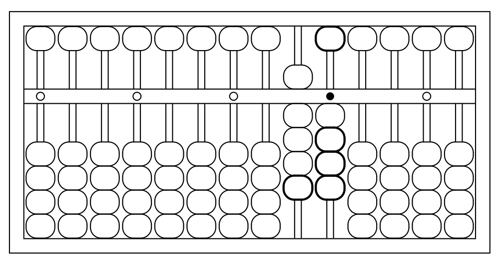


+30

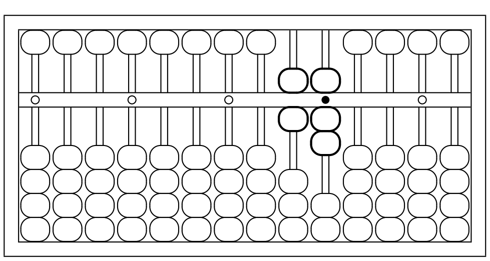
For this calculation, set 56 first. Since the tens-rod is the highest-valued rod of 38, add 3 to the tens-rod next. Since there are three 1-beads available at the bottom of the tens-rod, just move these beads up with your thumb. Execute $8 = -2 + 10$ in two parts. Use the fact $-2 = 3 - 5$ for -2 while observing the fifth rule of finger motion. This gives an intermediate result of 84.

Since there is a 1-bead available at the bottom of the tens-rod, just push it up with your thumb to add 10 and you are finished. The answer is 94.

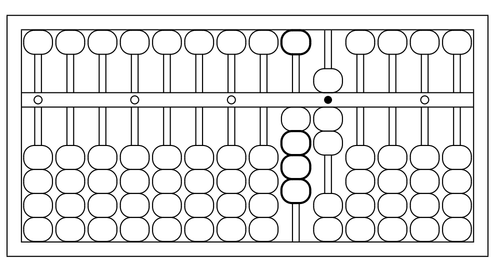


$+8 = 94$

$$\boxed{67 - 28 = 39} \qquad (6.21)$$



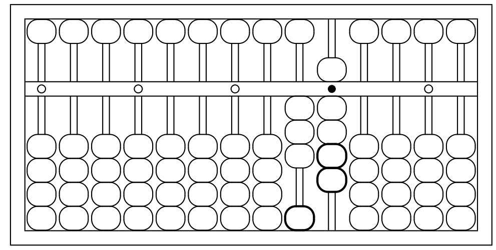
67



-20

For this calculation, set 67 first. Since the tens-rod is the highest-valued rod of 28, subtract 2 to the tens-rod next. Since there are not two 1-beads available for subtraction on the tens-rod, use the fact $-2 = 3 - 5$ while observing the fifth rule of finger motion. The intermediate result is 47.

Now subtract 8; use the fact $-8 = -10 + 2$ here. The -10 is easy since there is a 1-bead available for subtraction on the tens-rod. The $+2$ is also easy since there are two 1-beads available for addition on the ones-rod. Execute $-8 = -10 + 2$ while oberving the seventh rule of finger motion. The answer is 39.

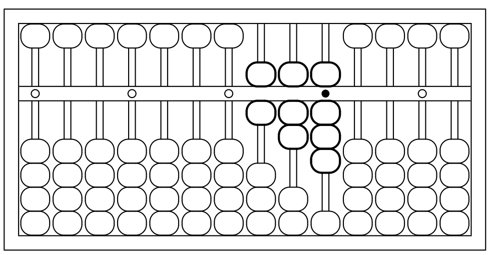


$-8 = 39$

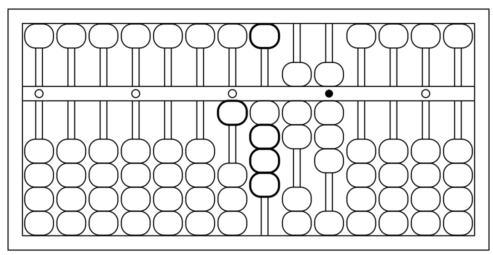
6.1.3 Adding and Subtracting Numbers Over Two Digits

Adding and subtracting numbers over two digits extends what you have already done with two-digit numbers. Note that earlier, you learned that there are four skills needed for addition and subtraction.[47] The following multi-digit examples are specially chosen so that only the first two skills — mastery of the facts for five and ten, and knowing which facts to apply to the task at hand — are exercised. You will add the third skill — using the stack — in Lesson 6.1.4. You will add the fourth skill — handling negative numbers — in Lesson 6.3.

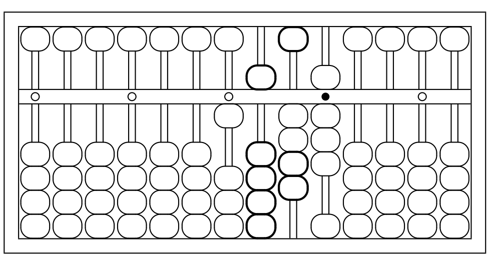
$$\boxed{678 + 876 = 1,554} \tag{6.22}$$



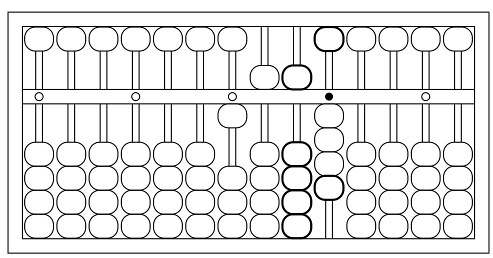
678



+800



+70



$+6 = 1,554$

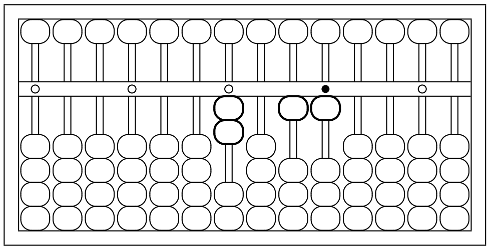
For this calculation, set 678 first. Since the hundreds-rod is the highest-valued rod of 876, add 8 to the hundreds-rod next. Since there are not a 5-bead and three 1-beads available for addition on the hundreds-rod, use the fact $8 = -2 + 10$ (and $-2 = 3 - 5$) while observing the fifth and sixth rules of finger motion. The intermediate result is $1,478$.

Now add 7 to the tens rod. Since there are not a 5-bead and two 1-beads available for addition on the tens-rod, use the fact $7 = -3 + 10$ (and $-3 = 2 - 5$) while observing the fifth and sixth rules of finger motion. The $+1$ on the hundreds-rod is not so easy since there is no 1-bead available for addition because all four 1-beads are already set. Use the fact $1 = 5 - 4$ on the hundreds-rod while observing the fourth rule of finger motion. The intermediate result is $1,548$.

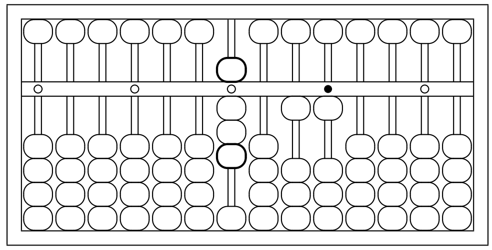
[47] See Lesson 6.1.

Finally, there is a $+6$ to do on the ones-rod. Since there are not a 5-bead and a 1-bead available for addition on the tens-rod, use the fact $6 = -4 + 10$ (and $-4 = 1 - 5$) while observing the fifth and sixth rules of finger motion. Once again, the $+1$ on the tens-rod is not so easy since there is no 1-bead available for addition because all four 1-beads are already set. Use the fact $1 = 5 - 4$ on the tens-rod while observing the fourth rule of finger motion. The answer is $1,554$.

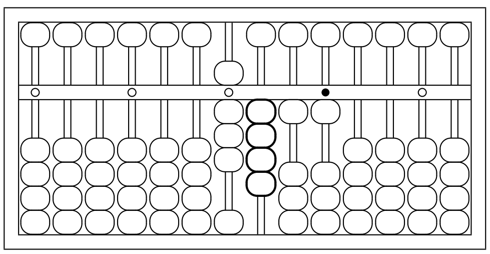
$$\boxed{2,011 + 6,429 = 8,440} \tag{6.23}$$



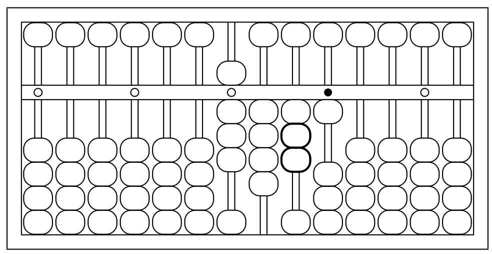
$2,011$



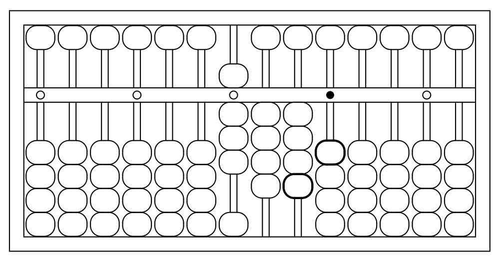
$+6,000$



$+400$



$+20$



$+9 = 8,440$

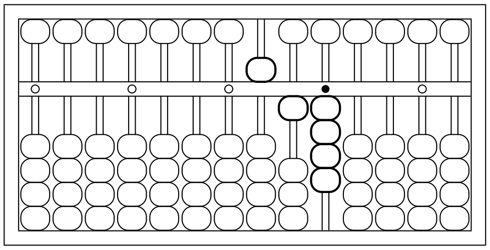
For this calculation, set $2,011$ first. Since the thousands-rod is the highest-valued rod of $6,429$, add 6 to the thousands-rod next. There are a 5-bead and a 1-bead available for addition on the thousands-rod, so just squeeze these toward the center bar while observing the second rule of finger motion. The intermediate result is $8,011$.

Now add 4 to the hundreds rod. There are four 1-beads available for addition on the hundreds-rod, so just push these toward the center with your thumb. The intermediate result is $8,411$.

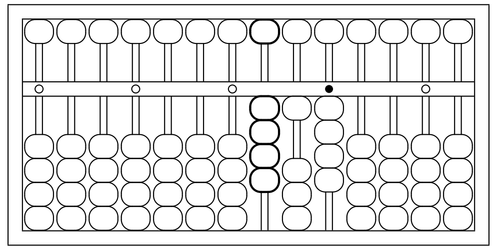
Now add 2 to the tens rod. There are two 1-beads available for addition on the tens-rod, so just push these toward the center with your thumb. The intermediate result is $8,431$.

Finally, there is a $+9$ to do on the ones-rod. Since there are not a 5-bead and four 1-beads available for addition on the ones-rod, use the fact $9 = -1 + 10$ while observing the sixth rule of finger motion. The answer is $8,440$.

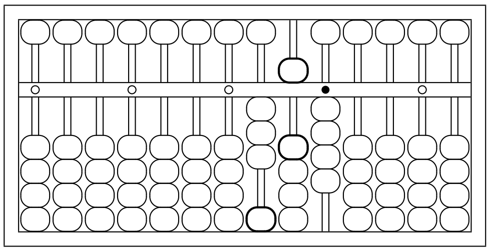
$$\boxed{514 - 165 = 349} \tag{6.24}$$



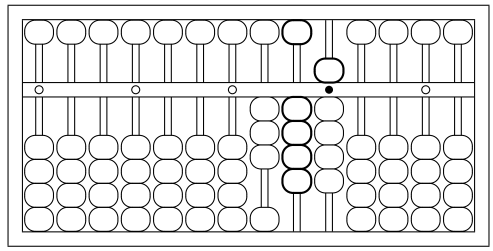
514



-100



-60



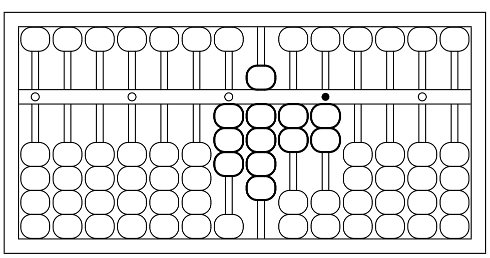
$-5 = 349$

For this calculation, set 514 first. Since the hundreds-rod is the highest-valued rod of 165, subtract 1 from the hundreds-rod next. Since there is no 1-bead available for subtraction on the hundreds-rod, use the fact $-1 = 4 - 5$ while observing the fifth rule of finger motion. The intermediate result is 414.

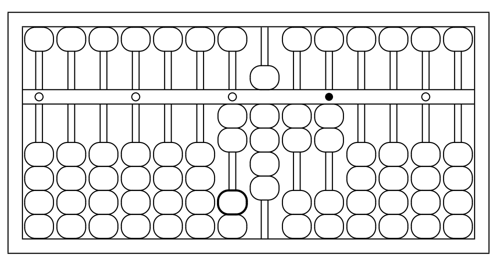
Now subtract 6 from the tens rod. Since there are not a 5-bead and a 1-bead available for subtraction on the tens-rod, use the fact $-6 = -10 + 4$ (and $4 = 5 - 1$) while observing the seventh and fourth rules of finger motion. The intermediate result is 354.

Finally, there is a -5 to do on the ones-rod. Since there is no 5-bead available for subtraction on the ones-rod, use the fact $-5 = -10 + 5$. The -1 on the tens-rod is not so easy since there is no 1-bead available for subtraction since none of the four 1-beads are set. Use the fact $-1 = 4 - 5$ on the tens-rod while observing the fifth rule of finger motion. To complete the fact for -5, add 5 on the ones rod by pulling a 5-bead down with your index finger. Since you worked on the tens-rod before pulling the 5-bead down on the ones-rod, you followed the seventh rule of finger motion. The answer is 349.

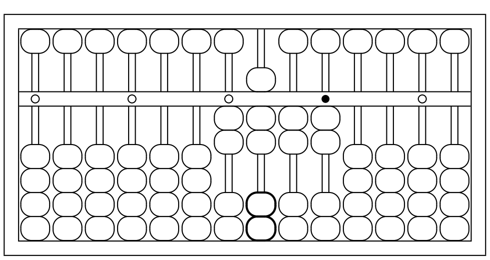
$$\boxed{3,922 - 1,258 = 2,664} \tag{6.25}$$



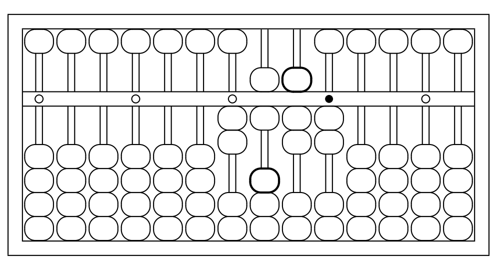
$3,922$



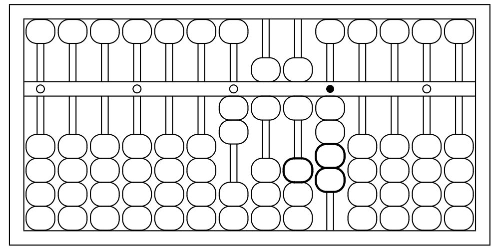
$-1,000$



-200



-50



$-8 = 2,664$

For this calculation, set $3,922$ first. Since the thousands-rod is the highest-valued rod of $1,258$, subtract 1 from the thousands-rod next. There is a 1-bead available for subtraction on the thousands-rod, so just pull it away from the center bar with your index finger. The intermediate result is $2,922$.

Now subtract 2 from the hundreds rod. There are two 1-beads available for subtraction on the hundreds-rod, so just pull these away from the center with your index finger. The intermediate result is $2,722$.

Now subtract 5 to the tens rod. Since there is no 5-bead available for subtraction from the tens-rod, use the fact $-5 = -10 + 5$ while observing the seventh rule of finger motion. The intermediate result is $2,672$.

Finally, there is a -8 to do on the ones-rod. Since there are not a 5-bead and three 1-beads available for subtraction from the ones-rod, use the fact $-8 = -10 + 2$ while observing the seventh rule of finger motion. The answer is $2,664$.

6.1.4 Using the Stack

Using the stack is the third skill for adding and subtracting on the abacus.[48] You are already familiar with how a stack works. Think of a stack of plastic cups sitting on a table. To add cups to the stack, add them to the top. To look at all the cups one-by-one, take cups off the top one after the other until you have reached the bottom cup.

For some addition and subtraction problems requiring more than one rod on the abacus, you create a stack of numbers that works just like a stack of cups. When you choose a fact to apply,[49] you may want to add or subtract a number from the rod to the left of the one you are working on. In some cases, you can't do this without adding or subtracting a number from the rod two rods to the left, perhaps also three rods to the left, and so on.[50] *The stack keeps track of what changes to make to all these rods.* With practice, stack thinking becomes second-nature.

For an example of stack thinking with addition, let's work the following problem:

$$\boxed{12 + 80 + 9 = 101} \tag{6.26}$$

The $12 + 80 = 92$ is easy, but how do you add 9 to 92? By building a stack and then disassembling it when it's "ready". *The stack is "ready" when its top number can be safely added or subtracted from the necessary rod on the abacus.* Our stack begins with

$$\llcorner +9 \lrcorner$$

The stack is not ready because we cannot simply add 9 to 92 since there are already two 1-beads set on the ones rod. So, we'll use the appropriate fact for this situation: $9 = -1 + 10$.[51] This changes the stack to

$$\begin{array}{c} \llcorner -1 \lrcorner \\ \llcorner +10 \lrcorner \end{array}$$

Looking at the top of the stack, we can do the -1, so the stack is ready. Subtract one from the ones rod and pop the -1 off the stack. This gives an intermediate result of 91. This reduces the stack to

$$\llcorner +10 \lrcorner$$

---

[48] See Lesson 6.1.

[49] See Lesson 6.1.1.

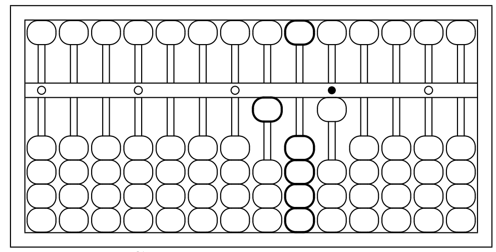
[50] If you learned how to do arithmetic before without an abacus, you used *carrying* with addition and *borrowing* with subtraction when faced with these kinds of situations.

[51] You should remember this fact readily. If not, play the Five and Ten Memory Game again. Continue to use the same facts for five and ten as you use for calculations that do not require a stack.

Is the stack ready again? No, because we cannot simply add a 1 to the tens-rod because there's a 9 on it. So we change that 1 to $-9+10$. This changes the stack to

$$\begin{array}{c} \llcorner -90 \lrcorner \\ \llcorner +100 \lrcorner \end{array}$$

Is the stack ready? Yes, because we can do the -90. So, subtract 9 from the tens rod and pop the -90 from the stack. There is a 100 still on the stack. Is the stack ready? Yes, because we can add a 1 to the hundreds rod. This pops the last number from the stack. The stack is empty and we have our final answer, 101.



Starting with 91,
pop -90, then pop $+100$

Note that for a stack of cups standing on a table, you can only take cups off from the top, not from anywhere else. It is the same with our stack of numbers. Removing a number off the top of the stack is called *popping* a number off the stack. Putting a number on top of the stack is called *pushing* a number onto the stack.[52]

From the previous example, you can see that the stack is a mental device that keeps track of changes across rods when a simple one- or two-rod calculation is not possible. *Keep to the rules of a stack — always push numbers on in the right order and always pop numbers from the top.*

For an example of stack thinking with subtraction, let's work the following problem:

$$\boxed{12+80+9+1902-5=1998} \tag{6.27}$$

We already did $12+80+9=101$. Adding 1902 is simple, giving a result of 2003. But how do you subtract 5 from 2003? By building a stack and then disassembling it when it's ready. Our stack begins with

$$\llcorner -5 \lrcorner$$

Since there is not a 5-bead available for subtraction on the ones-rod,we change -5 to $-10+5$. The stack is now

$$\begin{array}{c} \llcorner -10 \lrcorner \\ \llcorner +5 \lrcorner \end{array}$$

We can't do -1 on the tens-rod, so we change -1 to $-10+9$. This changes the stack to

$$\begin{array}{c} \llcorner -100 \lrcorner \\ \llcorner +90 \lrcorner \\ \llcorner +5 \lrcorner \end{array}$$

We can't do -1 on the hundreds-rod either, so we change -1 to $-10+9$ again. This changes the stack to

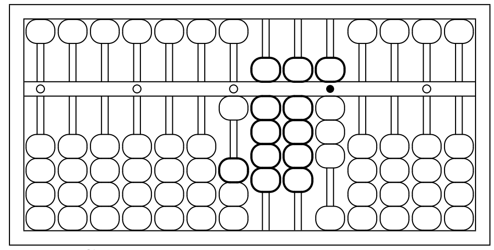
$$\begin{array}{c} \llcorner -1000 \lrcorner \\ \llcorner +900 \lrcorner \\ \llcorner +90 \lrcorner \\ \llcorner +5 \lrcorner \end{array}$$

[52] Another term describing such a structure of numbers is *Last-In-First-Out*, or LIFO. This is because the first number popped from the stack is always the one that was last pushed onto the stack.

We can do -1 on the thousands-rod, so the stack is finally ready. Actually, we can also do all the other numbers beneath it, so we pop them in order: -1000, 900, 90, and 5. What we have left on the stack after this is the answer, 1998.

In unusual circumstances, stack thinking must be used over and over when solving addition problems, and it takes some practice getting used to this.

Consider this unusual case for addition.



Starting with 2003,
pop -1000, $+900$, $+90$, and $+5$

$$29,999 + 1$$

Referring to the previous example of $92 + 9 = 101$, try to properly use the stack to perform $29,999 + 1$.

In unusual circumstances, the stack can grow to be quite large when solving subtraction problems, and it takes some practice to get used to this.

Consider this unusual case for subtraction.

$$30,000 - 1$$

Referring to the previous example of $2003 - 5 = 1998$, try to properly use the stack to perform $30,000 - 1$. The stack for this problem grows many levels deep. With practice, building and then popping such stacks will become second-nature.

6.2 An Excellent Exercise

The exercise is to add $123,456,789$ to itself nine times. The answer is $1,111,111,101$. Now, starting with the answer, subtract $123,456,789$ nine times. You should end up with zero.

By mastering this exercise, you will master the first three skills of addition and subtraction listed at the beginning of Lesson 6.1.

6.3 Negative Numbers

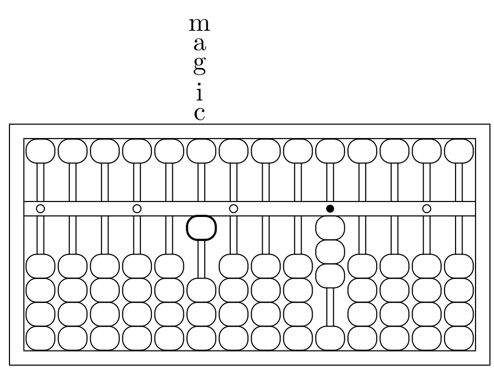
Handling negative numbers is the fourth and final skill for adding and subtracting on the abacus.[53] Can an abacus handle negative numbers?[54] Yes it can! You will be impressed by how naturally you can add positive and negative numbers, one after the other, whether the total is positive or negative.

$$\boxed{3 - 4 = -1} \tag{6.28}$$

Set 3 on your ones-rod. Subtract 2. That's easy. Set 3 on your ones-rod again. Subtract 3. That's easy. Set 3 on your ones-rod again. Subtract 4. What can we do about this? We have nothing to borrow from the tens-rod or from any other rod to the left.

To continue our calculation, we need something set on a higher rod. Let's set it ourselves. Let's *pick* the ten-thousands-rod and set 1 on it. Since we've done something unusual by setting a 1 on the ten-thousands-rod, let's call this rod the *magic-rod* instead of the ten-thousands-rod. (You can choose any empty rod to the left of your work to be the magic-rod, but choose it far enough left from your work so that you cannot confuse the magic-rod with your "real" rods.)

With the magic-rod set, we can build a stack and continue with subtracting 4 from 3 on the ones-rod. Build a stack until the stack is ready. When the stack is ready, it will look like this:



Pick a magic rod.

[53] See Lesson 6.1.

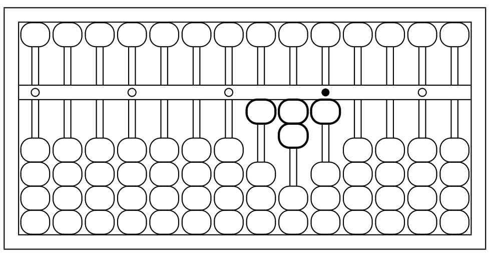
[54] If you love solving problems, close the workbook for a while and try to figure out how to handle negative numbers on the abacus. *Hint:* It works like integer arithmetic on computers.

$$\llcorner\ -10,000 \lrcorner$$
$$\llcorner\ +9,000 \lrcorner$$
$$\llcorner\ +900 \lrcorner$$
$$\llcorner\ +90 \lrcorner$$
$$\llcorner\ +6 \lrcorner$$

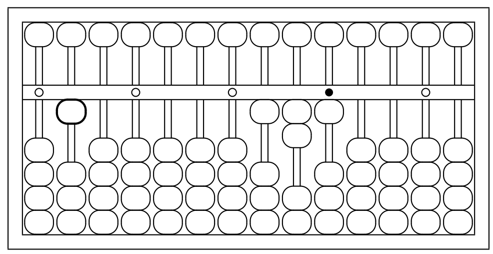
After you have popped all the numbers from the stack, you have your answer, 9,999. Wait a minute! 9,999 is not a negative number, or is it? It *is* a negative number because the magic rod is not set. If the magic rod were set, you would know that the answer was a positive number. See Lesson 6.3.1 to learn how to convert this number to something easier to read.

Let's do a few more examples.

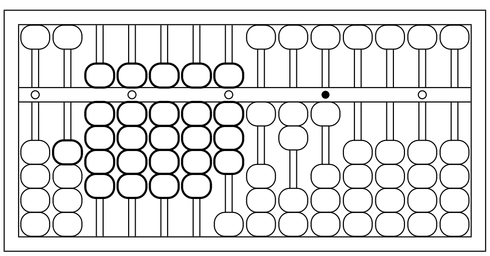
$$\boxed{121 - 2664 + 419 = -2124} \tag{6.29}$$



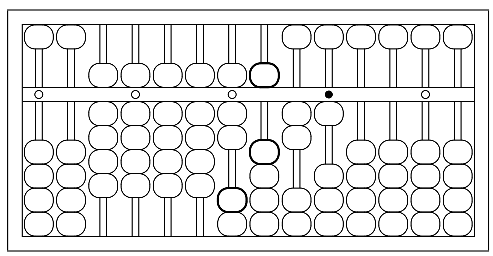
121



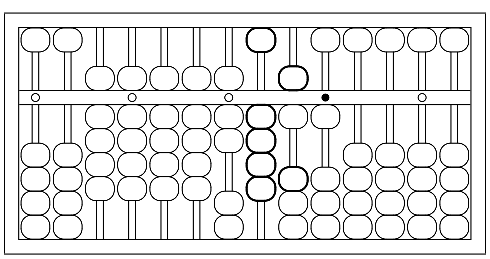
Pick magic rod.



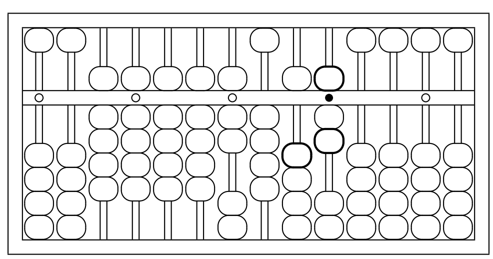
$-2,000$



-600



-60



-4

Start by setting 121. To subtract 2,664, begin by subtracting 2 on the thousands-rod. There is nothing set on the thousands-rod. If there were something set on any rod left of the thousands-rod, it would be possible to subtract 2 from the thousands-rod by building and then popping a stack.[55] However, there is nothing set left of the thousands-rod. In this case, choose a magic rod. Any rod to the left may be chosen as a magic rod, but if it is not chosen sufficiently far left, the magic rod may become confused with a "real" rod in the calculation. In this example, the 100-millions-rod if far left

[55]See Lesson 6.1.4.

and is chosen to be the magic rod. Now go ahead and do a -2 on the thousands-rod. Build a stack until it is ready. When ready, top of the stack will be:

$$\begin{array}{c}
\llcorner\ -1 \text{ on the } 100-\text{millions}-\text{rod}\ \lrcorner \\
\llcorner\ +9 \text{ on the } 10-\text{millions}-\text{rod}\ \lrcorner \\
\llcorner\ +9 \text{ on the millions}-\text{rod}\ \lrcorner \\
\llcorner +9 \text{ on the } 100-\text{thousands}-\text{rod} \lrcorner \\
\llcorner\ +9 \text{ on the } 10-\text{thousands}-\text{rod} \lrcorner \\
\llcorner\ +8 \text{ on the thousands}-\text{rod}\ \lrcorner
\end{array}$$

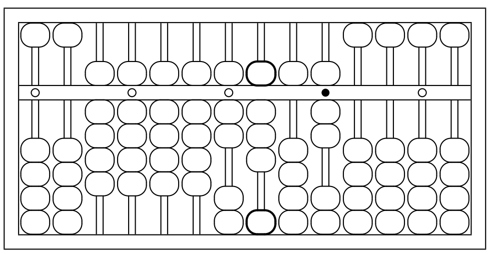
At this point, all the numbers can be popped from the stack, yielding an intermediate result $99,998,121$, and a 2 has been successfully subtracted from the thousands-rod.

Now subtract 6 from the hundreds-rod. This is done easily with a combination of the facts $-6 = -10+4$ and $4 = 5-1$. The intermediate result it $99,997,521$.

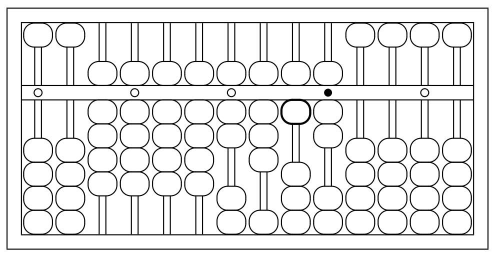
Now subtract 6 from the tens-rod. This is done easily with a combination of the facts $-6 = -10+4$, $-1 = 4-5$, and $4 = 5-1$. The intermediate result it $99,997,461$.

Now subtract 4 from the units-rod. This is done easily with the facts $-4 = -10+6$. The intermediate result is $99,997,467$.

There is still one more number to add. Add 419.

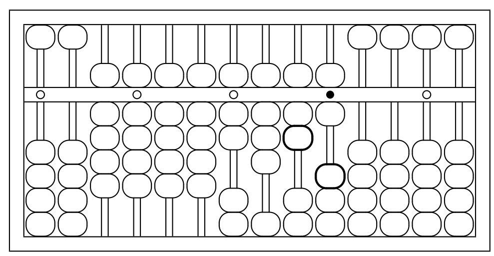


+400



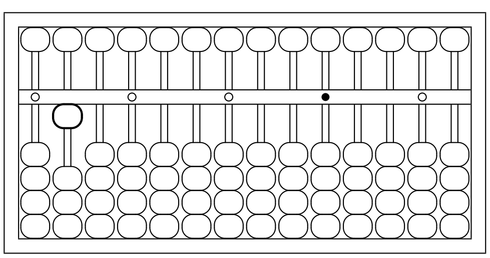
+10

To add 419, first add 4 to the hundreds rod. The fact to use here is $4 = 5-1$. The next step is to add 1 to the tens-rod. In this case, adding 1 is trivial — just move up a 1-bead with your thumb. The last step is to add 9 on the ones-rod. The fact to use here is $9 = -1+10$. The final result is $99,997,876$. This is a negative number because the magic rod is not set. See Lesson 6.3.1 to learn how to convert this number to something easier to read.

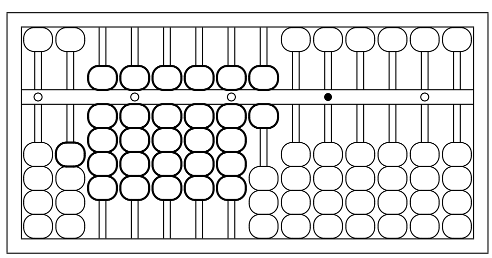


$+9 = 99,997,876$

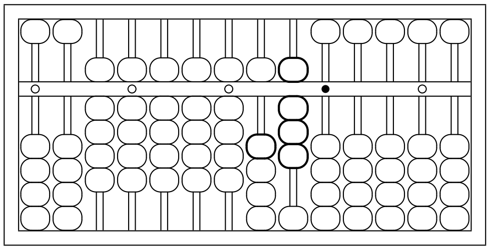
$$\boxed{-422+8400-8671=-693} \tag{6.30}$$



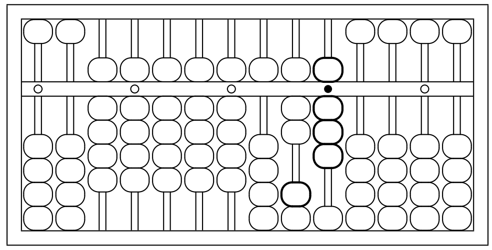
Pick a magic rod.



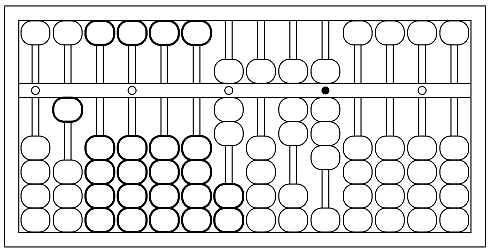
-400



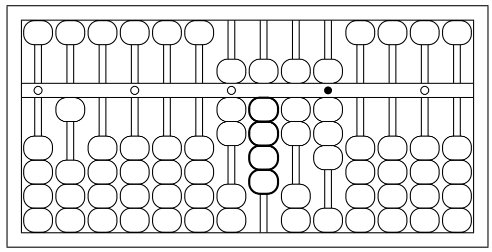
-20



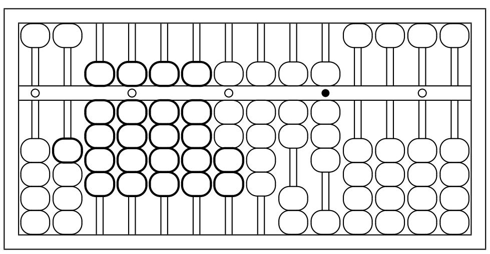
-2



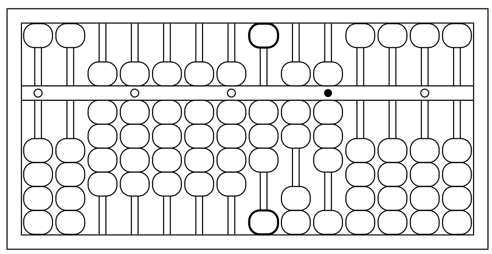
$+8,000$



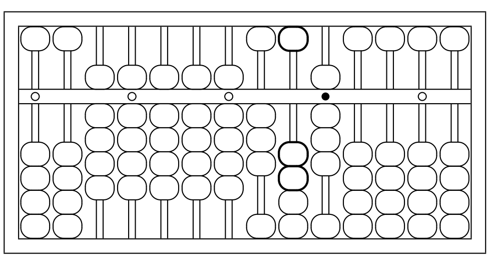
$+400$



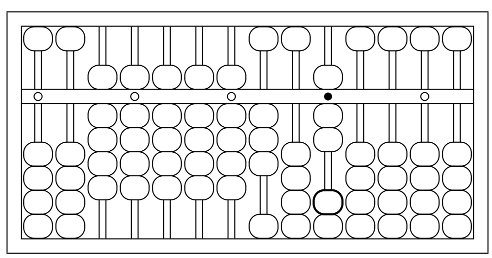
$-8,000$



-600



-70



-1

Start by setting -422; think of setting it in parts. First, set -400. To do this, you must immediately choose a magic rod[56] and set it. Build a stack up to the magic rod and then pop the numbers off the stack.[57] You will be left with a 6 on the hundreds-rod and several rods to the left with 9 set on them, up to but not including the magic rod because the magic rod was unset when the numbers were popped off the stack. Continuing with -422, now subtract 20. Since there is a 1-bead available on the hundreds-rod, use the fact $-2 = -10 + 8$ and apply it to the tens-rod. To finish with -422, now subtract 2. Since there is a 1-bead available on the tens-rod, use the fact $-2 = -10 + 8$ and apply it to the units-rod. Compare your results to those in the above diagrams.

[56] Choose the magic rod to be far left of the main working rods of the calculation. Choose one that "works best for you." In the following calculations that go into negative numbers, a magic rod has been *chosen* for each calculation. The discussion in the text reflects the choice. You may choose a magic rod different from one shown in a diagram, just choose it far enough left so as not to confuse your calculation once you run into large negative or large positive numbers.

[57] See Lesson 6.1.4.

Now add 8, 400. Adding 8 to the thousands-rod starts with the fact $8 = -2 + 10$ and requires building a new stack and popping numbers until all the 9-s have disappeared and the magic rod is set again.[58] Compare your results to those in the above diagrams.

Now subtract 8, 671. Subtracting 8 from the thousands-rod starts with the fact $-8 = -10 + 2$ which requires building a new stack until it reaches the magic rod and popping numbers so that the magic rod is unset and a row of 9-s reappears. Subsequently subracting 671 is straightforward, requiring only trivial facts — $-6 = -6$, $-7 = -7$, and $-1 = -1$ — for the computation. Compare your results to those in the above diagrams.

The final result is 99, 999, 307. This is a negative number because the magic rod is not set. See Lesson 6.3.1 to learn how to convert this number to something easier to read.

6.3.1 Decoding a Negative Result

To decode a negative result, start at the leftmost rod and work to the right. At each rod, subtract the number on the rod from nine. Set the result of the subtraction on the same rod. Finally, on the last (rightmost) rod only, add one. It is a simple procedure, once you are used to it.[59]

The result for Problem 6.28 was 9, 999, where the magic rod had been the 10-thousands-rod. Let's now decode it. From left to right, subtract 9 from every rod. The result is 0. Finally, add one to the last rod. The result is 1; therefore, the number 9, 999 stands for -1.

The result for Problem 6.29 was 99, 997, 876, where the magic rod had been the 100-millions-rod. Let's now decode it. From left to right, subtract 9 from every rod. The result is 2, 123. Finally, add one to the last rod. The result is 2, 124; therefore, the number 99, 997, 876 stands for $-2,124$.

The result for Problem 6.30 was 99, 999, 307, where the magic rod had been the 100-millions-rod. Let's now decode it. From left to right, subtract 9 from every rod. The result is 692. Finally, add one to the last rod. The result is 693; therefore, the number 99, 999, 307 stands for -693.

6.4 Changes in Clock Time

You learned in Lesson 3.2 about composing higher-value units with regard to keeping time. You noted that the rate of composing an hour-bead from ten-minute-beads is 6. You can use this fact and abacus-thinking to calculate changes in clock time.

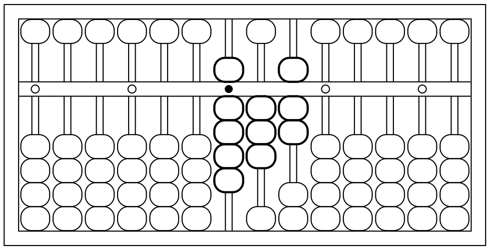
Since the rate of composing an hour-bead from ten-beads is 6, not 10, think of a *new principal number when calculating between the hour-rod and the ten-minute-rod.* This new principal number, of course, is 6. The numbers that have complements with six are 0, 1, 2, 3, 4, 5, and 6. Here is a model of the facts for six which can be used for clock calculations, with "one hour" being said instead of "six".

1. "Four plus two is one hour." ($4 + 2 = hour$)
2. "Two plus four is one hour." ($2 + 4 = hour$)
3. "Four is minus two plus one hour." ($4 = -2 + hour$)
4. "Minus four is minus one hour plus two." ($-4 = -hour + 2$)
5. "Two is minus four plus one hour." ($2 = -4 + hour$)
6. "Minus two is minus one hour plus four." ($-2 = -hour + 4$)

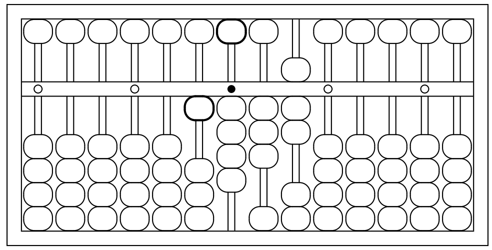
The pattern of these facts fit the pattern for the facts for principal number 10; that is, for the last four facts, state a negative number after saying the word "is".[60]

Suppose you put a turkey in the oven at 9:37 a.m. The turkey will be done in 5 hours and 46 minutes. What time will the turkey be done? Set 9:37 on your abacus and add 5:46. Use a principal number 6 when going between the ten-minute rod and the hour-rod. The result is 15:23, which is in "military" time. Subtract 12 to get regular time. The turkey will be done at 3:23 p.m.

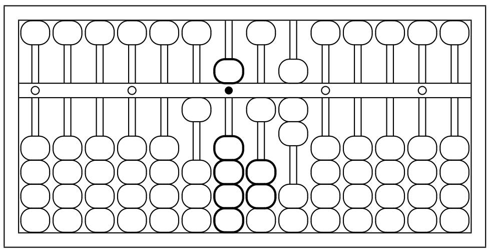
[58] See Lesson 6.1.4.
[59] This is how computers handle negative integers.
[60] This point is related to the discussion in Lesson 4.3.



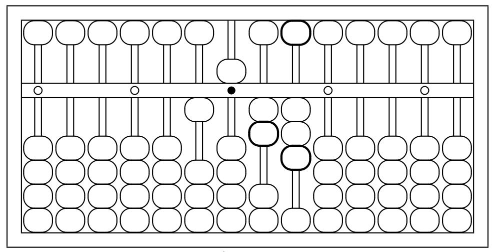
Set 9 : 37



Add 5 hours

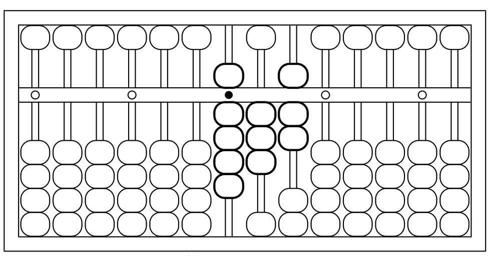


Add 40 minutes — principal number is 6

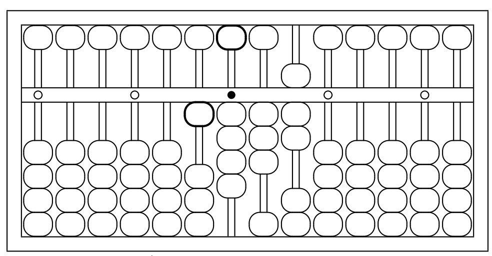


Add 6 minutes

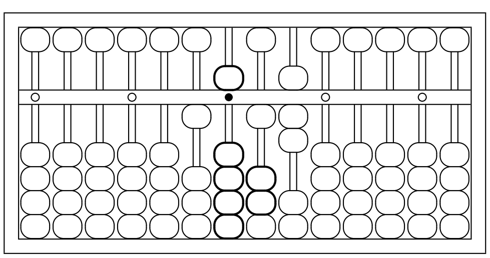
Suppose a baseball game begins at 11:55 a.m. and ends at 2:18 p.m. How long did the ball game last? First add 12 to 2:18 to convert the end time to military format — the game ended at 14:18. To find out how long the game lasted, set 14:18 on your abacus and subtract 11:55. Use a principal number 6 when going between the ten-minute rod and the hour-rod. The game lasted 2:23, or 2 hours and 23 minutes.



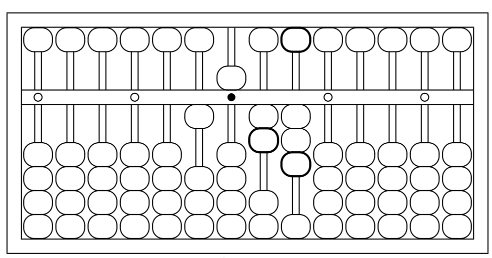
Set 9 : 37



Add 5 hours



Add 40 minutes — principal number is 6



Add 6 minutes

6.5 Exercises

Exercise 6.1

On the abacus, add the following columns of numbers in the order given.

10	1	4	7	4	0
8	5	2	5	8	8
18	6	6	12	12	8

Exercise 6.2

On the abacus, add the following columns of numbers in the order given.

98	11	83	86	42	98
30	75	47	50	1	81
128	86	130	136	43	179

Exercise 6.3

On the abacus, add the following columns of numbers in the order given.

84	−7	53	31	79	17
61	14	60	−30	−52	97
84	45	31	29	86	90
64	63	48	67	30	46
293	115	192	97	143	250

Exercise 6.4

On the abacus, add the following columns of numbers in the order given.

18	0	50	16	53	12,955
66	69	74	17	86	−43
36	5	87	69	2	47
1	29	21	71	83	68
97	63	67	6,027	2	98,666
−10	−34	7	15	66	−15
208	132	306	6,215	292	111,678

Exercise 6.5

On the abacus, add the following columns of numbers in the order given.

75.03	9.09	76,019.28	86,608.88
32.00	0.80	3.32	43,078.61
62.64	7.43	97.62	21.03
70.99	95,205.68	5.11	97.25
89.99	13.65	2.12	80.35
9.02	14.95	45,214.84	30.73
94,979.85	50.43	94,775.39	62,799.07
35.16	43.09	81.26	68.94
95,354.68	95,345.12	216,198.94	192,784.86

Exercise 6.6

Practice the "excellent exercise". Add all the way to $1,111,111,101$ and subtract back down to zero. Practice until you can complete the exercise, without error, in less than five minutes.

Exercise 6.7

On the abacus, add the following columns of numbers in the order given.

−7	−6	5	3	−10	2
−5	−8	−1	0	−2	8
−12	−14	4	3	−12	10

Exercise 6.8

On the abacus, add the following columns of numbers in the order given.

−5	−9	−2	1	0	−1
7	−1	3	−2	10	6
7	4	−2	−4	8	5
−4	−2	5	−2	−1	−8
−6	−5	−2	−1	−3	0
−4	−3	−2	−6	−7	0
2	−1	−5	1	6	−2
−5	3	−3	6	−1	5
−8	−14	−8	−7	12	5

Exercise 6.9

On the abacus, add the following columns of numbers in the order given.

43.11	70.41	−64,123.54	−20.13
17.88	−74.55	−23.53	−39.88
38.64	−99,465.95	85,226.44	−29,595.95
−74.55	−35.56	−11.38	−79.44
−9.61	94.05	−8.89	33.10
9.91	−82.33	−72.18	−79.25
−7.63	−15.67	−51.78	25,469.97
32.37	−44,760.14	88.32	11.65
50.12	−144,269.74	21,023.46	−4,299.93

Exercise 6.10

You are a police officer who has found a bomb hidden in a bus terminal. There is a digital clock attached to the bomb; it reads 10:46 and is counting forward in time. Assuming that the bomb will detonate when its clock reaches 12:00, how much time is there to warn others and evacuate the terminal? You look at your clock and it reads 6:49 p.m. What time will the bomb detonate? (Calculate on the abacus.)

You have 1:14 (1 hour and 14 minutes) to warn others and evacuate the terminal. The bomb will detonate at 8:03 p.m.

Exercise 6.11

Calculate on the abacus, in hours and minutes, how long the following runners took to complete an orienteering course. Determine who are the top three finishers.

Name	Start Time	End Time	Elapsed Time
Charese Gearhart-Dekrean	8:06	9:41	1:35
Grace Cho	8:16	9:10	0:54
Eric Williamson	7:54	9:49	1:55
Marja Kankaaranta	7:56	8:52	0:56
Steve Kiley	8:14	9:07	0:53
Jim Knapp	7:48	9:20	1:32
Azamat Mantgar	8:20	9:24	1:04

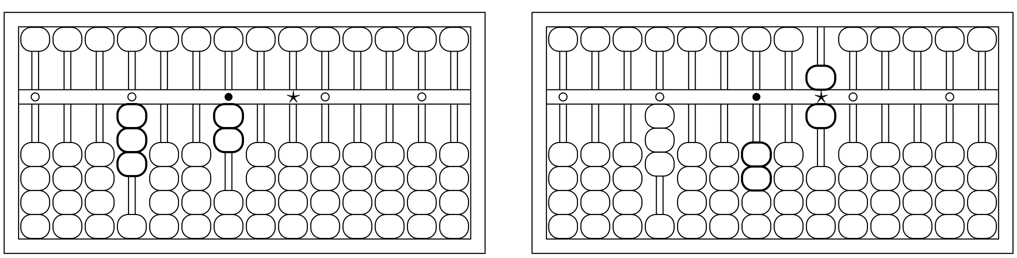
The top three finishers are Kiley, Cho, and Kankaaranta.

7 Multiplication

Multiplication on the abacus first requires mastery of the multiplication table up to 9 × 9. After that, multiplication involves simple multiplication of all digits in the multiplier with all the digits in the multiplicand. The hardest part is to correctly determine the units rod for the answer and to add the product of two numbers to the correct answer rod on the abacus.

Set the multiplier near the center of your abacus. Leave two empty rods to the left of the multiplier; then set the multiplicand to the left of the multiplier.

$$\boxed{3 \times 2 = 6} \tag{7.1}$$



3 × 2 = 6

First, set up the multiplier on the abacus—multiplier on the right, multiplicand on the left. (Think of the multiplicand as the number you start with and the multiplier as how many times you multiply the multiplicand.) Always set the multiplier on a units rod, as you would normally set a number for addition or subtraction. Once you have set the multiplier, leave two or more empty rods to its left and then set the digits of the multiplcand to the left of these empty rods. In this case the multiplicand 3 has been set to the left of the multiplier 2, with two empty rods between.

Now determine where the units-rod for your answer will be. *Determining and remembering which rod is the units-rod for the answer is the most error-prone part of multiplication and division on the abacus.* In this lesson, we are dealing only with multiplication.[61] The units-rod for the answer need not be a rod with a dot on it. (It may happen to be a rod with a dot on it, but often it is not.) *The*

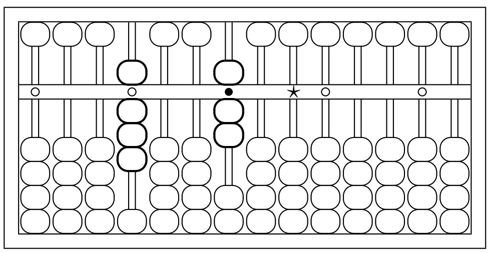
[61] See Lesson 8 to learn how to divide on the abacus.

highest-valued rod in the multiplicand determines where you will set the units-rod for the answer. If the highest-valued rod for the multiplicand is the units-rod, then the units-rod for the answer is two rods to the right of the units-rod of the multiplicand. For the examples in this workbook, the units-rod for the answer of a multiplication or division problem will be shown with a star ($\star$). Of course, the abacus on which you calculate will not have a star to show you which is the units-rod for the answer; it will be your job to determine which rod it is.

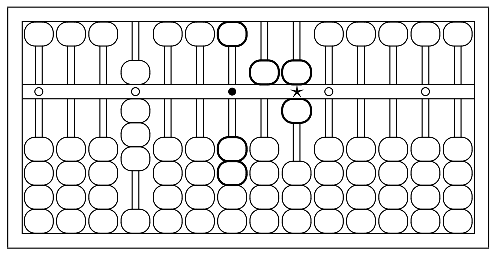
In this case, the multiplicand is 3, whose highest rod is the units-rod. So the units-rod for the answer is two rods to the right of the units-rod of the multiplier, as indicated by the star.

Now you are ready to multiply your digits. Multiply 3×2 and set the result two rods to the right of the 2.[62] It happens to be that you set this result on the units-rod for the answer. When you have set the 6, remove the 2 from the multiplier because you are now finished multiplying with it. The answer is 6.

$$\boxed{8 \times 7 = 56} \tag{7.2}$$



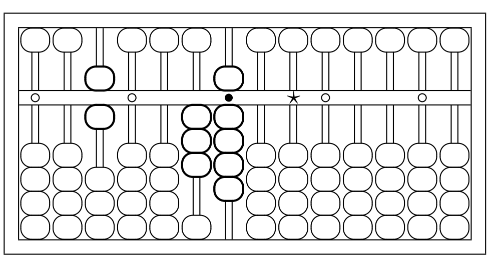
8×7



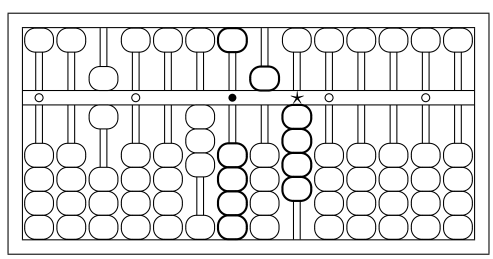
$= 56$

This example is almost the same as the previous one. The only difference is that there are two digits in the answer 56 while in Example 7.1 there was only one digit in the answer 6. Set the multiplier, 7, on a rod with a dot near the center of the abacus. Leave two empty rods to the left and then set the multiplicand, 8. Next, determine which rod is the units-rod for your answer. Since the highest rod in 8, the multiplicand, is the ones-rod, the units-rod for the answer is two rods to the right of the units rod for 7, the multiplicand. (The unit-rod for the answer is shown by a $\star$.) When you set 56 "two rods to the right" of the 7, this means that the *units-rod* of 56 lies two rods to the right. Once you are finished multiplying the digit 7 in the multiplier, remove it from the abacus. The answer is 56.

$$\boxed{6 \times 39 = 234} \tag{7.3}$$

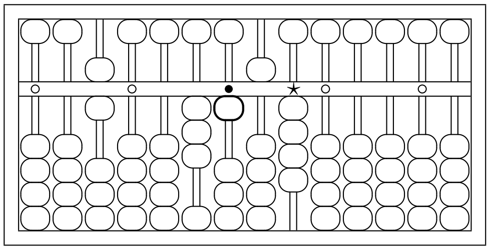


6×39

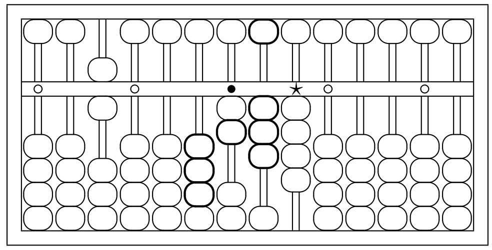


add $6 \times 9 = 54$ on the proper rods

[62] Why set this number two rods to the right of the 2? Because you need two rods to hold the largest possible number resulting from the multiplication of a single-digit number by a single-digit number, that number being the two-digit number 81.



$6 \times 3 = 18$; add the 1 on the proper rod



$6 \times 3 = 18$; add the 8 on the proper rod

Choose a rod with a dot near the center of the abacus and use it as the units-rod for 39. Set 39, the multiplier. Leave two empty rods to the left of the multiplier and set 6, the multiplicand.

Determine where the units rod will be for the answer. Since the highest-valued rod for the multiplicand is the ones-rod, the units-rod for the answer will be two rods to the right of the units-rod of the multiplier.

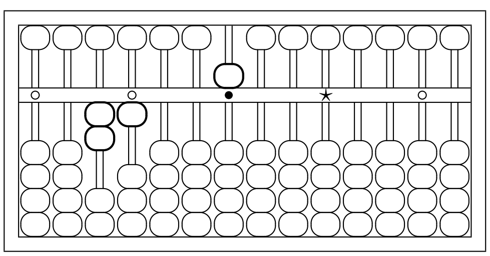
You are now ready to multiply the digits of the multiplicand by the digits in the multiplier. The answer will be set on the rods right of the multiplier. Which digit of 39 should you multiply first, 3 or 9? Technically, you could start with either and still get the right answer. But in order to use the least possible number of rods on the abacus, start with the lowest digit, 9.[63] $6 \times 9 = 54$. Set this result two rods to the right of the 9. After the result is set, you are finished with the 9, so remove it from your abacus. You have now reached the point depicted in the second of the four diagrams above.

Now multiply the 3 (tens digit of 39). $6 \times 3 = 18$. Set this result two rods to the right of the 3. First set the 1, which, being the tens-digit of 18, is set one rod to the right of the 3. You have now reached the point depicted in the third of the four diagrams above.

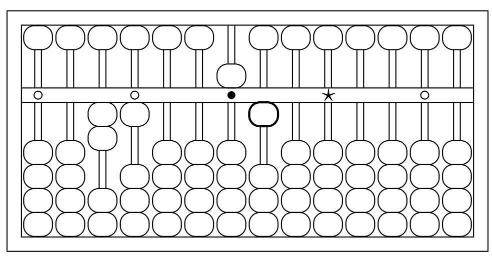
Finally add the 8 of 18 two rods to the right of the 3. To add 8 to this rod, you will need to employ facts of five and ten. After adding the 8, you are finished with the 3 in the multiplier, so remove it from your abacus.

Recall which is the units-rod for your answer and read the answer, 234.

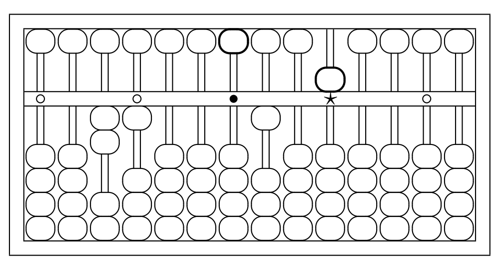
$$\boxed{21 \times 5 = 105} \tag{7.4}$$



21×5



$2 \times 5 = 10$; add the 10 on the proper rod



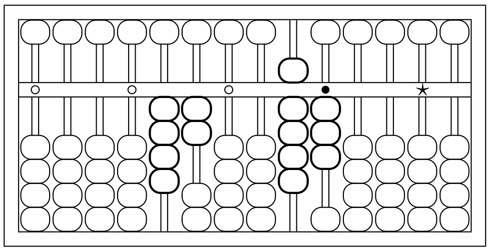
$1 \times 5 = 5$; add the 5 on the proper rod

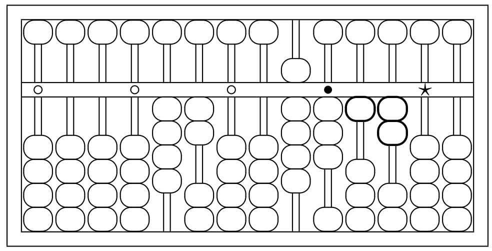
First set the multiplier, 5, on a rod with a dot. Leave two empty rods to the left of the multiplier and then set the multiplicand, 21. This is the first example of a multiplicand of more than one digit. As stated before, *the highest-valued rod in the multiplicand determines where you will set the units-rod for the answer.* This time, the highest-valued rod of the multiplicand is the tens-rod. In the previous examples, where the highest rod of the multiplicand was the *ones-rod*, the units-rod for the answer was set two rods to the right of the units-rod of the multiplier. In this problem the highest rod of the multiplicand is the tens-rod, which is one rod

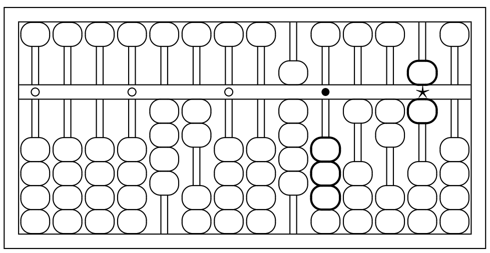
[63]For multiplying on paper, you are not interested in conserving abacus rods since you are no longer manipulating beads on an abacus. So on paper, start multiplying the highest digit, 3. See Example 9.10 in Lesson 9.2.

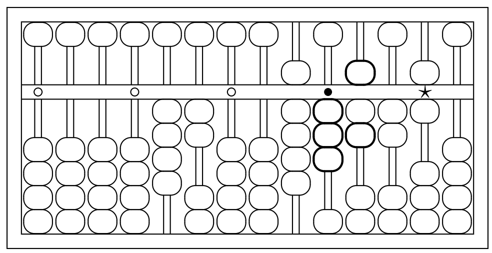
higher than the ones-rod. Therefore, instead of two rods to the right, the units-rod for the answer is now set *three* rods to the right of the units-rod of the multiplier.[64]

$$\boxed{42 \times 93 = 3,906} \tag{7.5}$$

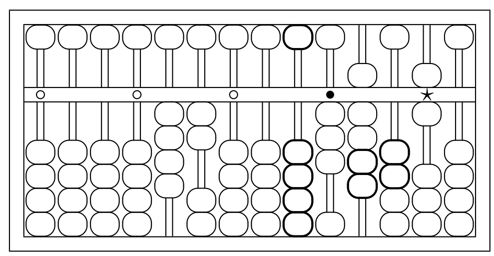

42×93


add $4 \times 3 = 12$ on the proper rods


add $2 \times 3 = 6$ on the proper rod


add $4 \times 9 = 36$ on the proper rods

First set the multiplier, 93, on a rod with a dot. Leave two empty rods to the left of the multiplier and then set the multiplicand, 42. *The highest-valued rod in the multiplicand determines where you will set the units-rod for the answer.* The highest-valued rod of the multiplicand is the tens-rod. Where the highest rod of the multiplicand was the *ones-rod*, the units-rod for the answer is set two rods to the right of the units-rod of the multiplier. But here, as in the previous example, the highest rod of the multiplicand is the tens-rod, which is one rod higher than the ones-rod. Therefore, instead of two rods to the right, the units-rod for the answer is *three* rods to the right of the units-rod of the multiplier.


add $2 \times 9 = 18$ on the proper rods

This is the first example with a multiplier of more than one digit. In this case, the multiplicand has two digits. The strategy for multiplication is to first multiply through each rod of the multiplicand for the lowest rod of the multiplier, then remove the number on the lowest rod of the multiplier. Repeat the procedure for the next highest rod of the multiplier, multiplying through each rod of the multipliand for this rod. Note that the order of rods to be considered is reversed between the multiplicand and the multiplier. When working through the digits of the multiplicand, start from the highest rod and work to the lowest rod. When working through the digits of the multiplier, start from the lowest rod and work to the highest rod.

In this example the lowest rod of the multiplier is 3. Keeping the 3 fixed, work through all the digits of the mulitplicand working from highest to lowest. The highest digit on the multiplicand is 4; $4 \times 3 = 12$, so add 12 on the rod that is two rods to the right of the 3. The next lowest digit on the multiplicand is 2; $2 \times 3 = 6$, so, moving one rod to the right, add 6 on the rod that is *three* rods to the

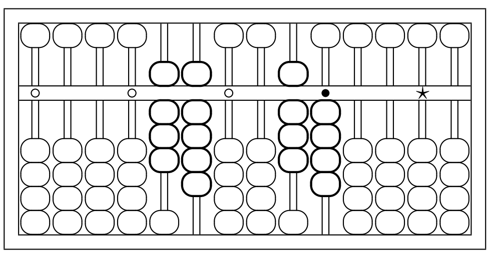
[64]If the highest rod of the multiplicand is the hundreds-rod, the units-rod for the answer is set *four* rods to the right of the units-rod of the multiplier. If the highest rod of the multiplicand is the thousands-rod, *five* rods to the right; ten-thousands-rod, *six* rods to the right; hundred-thousands-rod, *seven* rods to the right; etc. If the highest rod of the multiplicand is the *tenths-rod*, the units-rod for the answer is set *one* rod to the right of the units-rod of the multiplier; hundredths-rod, *zero* rods to the right (same rod); thousandths-rod, *one* rod to the *left*; ten-thousandths-rod, *two* rods to the left; hundred-thousandths-rod, *three* rods to the left; etc.

right of the 3.[65] There are no more digits left on the multiplicand, so you are now done with the 3 of the multiplier; so remove the 3 from the multiplier since you are finished with it.

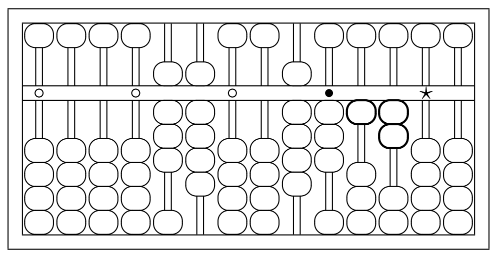
The *next highest rod of the multiplier* is 9. With the 3 having been cleared away, you are now clear to repeat the procedure with the 9. Keeping the 9 fixed, work through all the digits of the mulitplicand working, as always, from highest to lowest. The highest digit on the multiplier is 4; $4 \times 9 = 36$, so add 36 on the rod that is two rods to the right of the 9. At this point, there already is a number on this rod, a 1 from the previous multiplication of the digit 3. That is fine and normal; just add the 36 right on top of the 1, as shown in the diagram. The next lowest digit on the multiplicand is 2; $2 \times 9 = 18$, so add 18 on the rod that is *three* rods to the right of the 9. Here, you add 18 right on top of a 72 that is there from previous computation. That's right, go right ahead and add the 18; you will now have a 90 on the two rods that were once 72. There are no more digits left on the multiplicand, so you are now done with the 9 of the multiplier. Remove the 9 from the multiplier since you are finished with it. Since there are no more higher digits left in the multiplier, you are finished. The answer is $3,906$.

$$\boxed{89 \times 84 = 7,476} \tag{7.6}$$

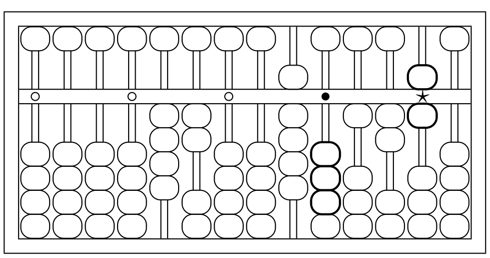
THESE DRAWINGS STILL NEED TO BE EDITED TO CORRECTLY PORTRAY THIS EXAMPLE.



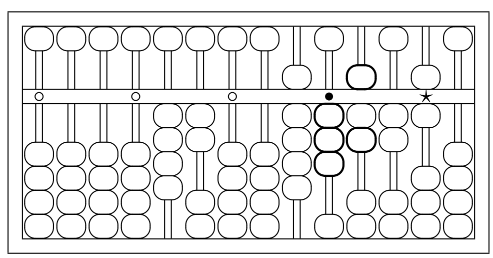
89×84



add $8 \times 4 = 32$ on the proper rods

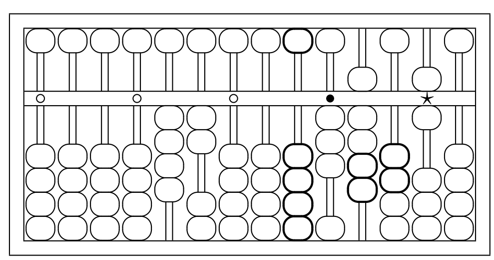


add $2 \times 3 = 6$ on the proper rod



add $4 \times 9 = 36$ on the proper rods

First set the multiplier, 84, on a rod with a dot. Leave two empty rods to the left of the multiplier and then set the multiplicand, 89. For the last time, *the highest-valued rod in the multiplicand determines where you will set the units-rod for the answer.* The highest-valued rod of the multiplicand is the tens-rod. Therefore, the units-rod for the answer is *three* rods to the right of the units-rod of the multiplier.



add $2 \times 9 = 18$ on the proper rods

Again, the multiplicand has more than one digit. The strategy for multiplication is to multiply through each rod of the multiplicand, highest to lowest, for the lowest rod of the multiplier, then

[65] It makes sense that the 6 be set one rod to the right of the rod where the 12 was set. The 12 came from multiplying the value of the tens-rod of the multiplicand (4) with the 3 of the multiplier. The 6 came from multiplying the value of the *ones*-rod of the multiplicand (2) with the 3 of the multiplier. Moving from the tens rod to the ones rod reduces the value of that product by a factor of ten, which on the abacus means to shift one rod to the right.

remove the number from lowest rod of the multiplier. Repeat the procedure for the next highest rod of the multiplier, multiplying through each rod of the multiplicand, highest to lowest, for this rod. Note that the order of rods to be considered is reversed between the multiplicand and the multiplier. When working through the digits of the multiplicand, start from the highest rod and work to the lowest rod. When working through the digits of the multiplier, start from the lowest rod and work to the highest rod.

In this example the lowest rod of the multiplier is 4. Keeping the 4 fixed, work through all the digits of the mulitplicand working from highest to lowest. The highest digit on the multiplicand is 8; $8 \times 4 = 32$, so add 32 on the rod that is two rods to the right of the 4.

CONTINUE EDITING TEXT FROM THIS POINT.

The next lowest digit on the multiplier is 2; $2 \times 3 = 6$, so add 6 on the rod that is *three* rods to the right of the 3.[66] There are no more digits left on the multiplier, so you are now done with the 3 of the multiplicand; so remove the 3 from the multiplicand since you are finished with it.

The *next highest rod of the multiplicand* is 9. With the 3 having been cleared away, you are now clear to repeat the procedure with the 9. Keeping the 9 fixed, work through all the digits of the mulitplier working, as always, from highest to lowest. The highest digit on the multiplier is 4; $4 \times 9 = 36$, so add 36 on the rod that is two rods to the right of the 9. At this point, there already is a number on this rod, a 1 from the previous multiplication of the digit 3. That is fine and normal; just add the 36 right on top of the 1, as shown in the diagram. The next lowest digit on the multiplier is 2; $2 \times 9 = 18$, so add 18 on the rod that is *three* rods to the right of the 9. Here, you add 18 right on top of a 72 that is there from previous work. That's right, go right ahead and add the 18; you will now have a 90 on the two rods that were once 72. There are no more digits left on the multiplier, so you are now done with the 9 of the multiplicand; so remove the 9 from the multiplicand since you are finished with it. Since there are no more higher digits left in the multiplicand, you are finished. The answer is $3,906$.

$$\boxed{44 \times 329 = 14,476} \tag{7.7}$$

$$\boxed{89 \times 703 = 62,567} \tag{7.8}$$

If the multiplicand and room for the answer do not leave enough space for the multiplier to be set on the left of the abacus, then do not set the multiplier at all. Instead, write the multiplier on a piece of paper and refer to it while multiplying just as you would if you had set it on the left of the abacus.

$$\boxed{35 \times 3,246,094 = 113,613,290} \tag{7.9}$$

7.1 Dealing With the Decimal Point

When deciding what will be the units rod for your answer, remember: *For multiplication, start two rods to the right; for division, start two rods to the left.* In this section, we will only deal with placing the decimal point for multiplication. Placing the decimal point for division is discussed in Lesson 8.1.

$$\boxed{30 \times 2 = 60} \tag{7.10}$$

$$\boxed{8 \times 0.7 = 5.6} \tag{7.11}$$

[66]It makes sense that the 6 be set one rod to the right of the rod where the 12 was set. The 12 came from multiplying the value of the tens-rod of the multiplier (4) with the 3 of the multiplicand. The 6 came from multiplying the value of the *ones*-rod of the multiplier (2) with the 3 of the multiplicand. Moving from the tens rod to the ones rod reduces the value of that product by a factor of ten, which on the abacus means to shift one rod to the right.

$$\boxed{0.6 \times 390 = 234} \tag{7.12}$$

$$\boxed{.21 \times 0.5 = 0.105} \tag{7.13}$$

$$\boxed{42,000 \times 930 = 39,060,000} \tag{7.14}$$

$$\boxed{0.000089 \times 8,400 = 0.7476} \tag{7.15}$$

$$\boxed{4.4 \times 3.29 = 14.476} \tag{7.16}$$

$$\boxed{890 \times 0.00703 = 6.2567} \tag{7.17}$$

All this works the same whether the numbers both fit on the abacus or whether only the multiplicand fits on the abacus.

$$\boxed{0.350 \times 3,246.094 = 1,136.13290} \tag{7.18}$$

7.2 Exercises

8 Division

Division, like multiplication, first requires mastery of the multiplication table up to 9×9.

$$\boxed{\tfrac{9}{3} = 3} \tag{8.1}$$

$$\boxed{\tfrac{35}{7} = 5} \tag{8.2}$$

$$\boxed{\tfrac{897}{3} = 299} \tag{8.3}$$

$$\boxed{\tfrac{1{,}166}{2} = 583} \tag{8.4}$$

Of course, not all division problems work out to an even integer. For these problems, decide you many significant figures you want for your answer. For Problem 8.5, round your answer to three significant digits. To round your answer to three sig figs, calculate to four digits and then round to three digits.[67]

$$\boxed{\tfrac{6{,}331}{7} \approx 904} \tag{8.5}$$

Now let's move on to divisors of two or more digits.

$$\boxed{\tfrac{3{,}400}{85} = 40} \tag{8.6}$$

[67] When rounding occurs, the answer is no longer exact. So, instead of using the = sign, now use the ≈ sign to indicate an approximate answer.

This one has an approximate answer. Once again, round to three sig figs.

$$\boxed{\frac{4{,}969}{43} \approx 116} \tag{8.7}$$

$$\boxed{\frac{52{,}020}{612} = 85} \tag{8.8}$$

This one has an approximate answer. Once again, round to three sig figs.

$$\boxed{\frac{7{,}015}{303} \approx 23.2} \tag{8.9}$$

8.1 Dealing With the Decimal Point

When deciding what will be the units rod for your answer, remember: *For multiplication, start two rods to the right; for division, start two rods to the left.* In this section, we will only deal with placing the decimal point for division. Placing the decimal point for multiplication is discussed in Lesson 7.1.

$$\boxed{\frac{9}{30} = 0.3} \tag{8.10}$$

$$\boxed{\frac{350}{0.7} = 500} \tag{8.11}$$

$$\boxed{\frac{0.00897}{3} = 0.00299} \tag{8.12}$$

$$\boxed{\frac{0.1166}{0.02} = 5.83} \tag{8.13}$$

$$\boxed{\frac{633{,}100}{700} \approx 904} \tag{8.14}$$

$$\boxed{\frac{340}{0.00085} = 400,000} \tag{8.15}$$

$$\boxed{\frac{4{,}969}{43{,}000} \approx 0.116} \tag{8.16}$$

$$\boxed{\frac{52.020}{612} = 0.085} \tag{8.17}$$

$$\boxed{\frac{701{,}500}{30.3} \approx 23,200} \tag{8.18}$$

8.2 Exercises

9 Calculating on Paper

9.1 Addition and Subtraction

Adding and subtracting on paper is different from adding and subtracting on the abacus. On the abacus, you add or subtract each number in its entirety, one after the other. On paper, you write all the numbers down first with the columns carefully lined up; then you sum each column from highest rod to lowest rod.

Lesson 10 describes mental abacus-based calculations as being done on an abacus you see in your mind. Adding and subtracting on paper does require that you do simple mental math. You must add and subtract columns of single-digit numbers in your head, which is easily done if you have sufficient practice adding and subtracting on the abacus.

The "standard" method of adding and subtracting on paper calculates from the lowest digit (rod) to the highest. One advantage of the abacus-based paper calculations is that since the calculation moves from highest rod to lowest, you have the option of choosing to make an estimate of the answer without completing the calculation, because after summing the first few high rods, you have a pretty good idea of what the values for the highest rods will be after the calculation is complete. In some cases, this is good enough and you need calculate no more. When calculating on paper, since you are working from low rod to high rod, you do not know what the values of the high rods will be until you have finished the entire calculation.

If anything negative is to be said about the abacus-based paper method, it is that it requires more paper than the "standard" method. On the other hand, it is easier to see and correct mistakes with the abacus-based method.

$$\boxed{3+1=4} \tag{9.1}$$

This is a trivial problem. Line up the 3 and the 1 on the ones column. Add them in your head. On the abacus, $3+1$ is trivial. There are no special facts to remember, just push up three 1-beads and then push up one more 1-bead. Draw a horizontal line under the column and report the answer.

•
3
1
4

$$\boxed{4+3=7} \tag{9.2}$$

Line up the 4 and the 3 on the ones column. Add them in your head. On the abacus, after setting 4, you will need the fact $3 = 5 - 2$ in order to add 3. Do this all in your head; moving the fingers on your right hand as if you were manipulating a real abacus will likely help you see the calculation clearly. Draw a horizontal line under the column and report the answer.

•
4
3
7

$$\boxed{3+5+8+8=24} \tag{9.3}$$

Line up the numbers on the ones column. Add them in your head. $3+5$ is easy, $8+8$ requires the use of the fact $8 = -2+10$, and $16+8$ requires the use of $8 = -2+10$ where $-2 = 3-5$. Doing this all in your head for the first time is not easy. But it will become easy with practice on both the abacus and in your head. Draw a horizontal line under the column and report the answer.

	•
	3
	5
	8
	8
2	4

$$\boxed{13+16+13+3=45} \tag{9.4}$$

Line up the numbers in their respective columns. Since the tens-column is the highest column with any numbers in it, add these numbers first. The result is 3. Draw a horizontal line and write a 3 in the tens-column. Now add the numbers in the ones-column. If you have difficulty adding these numbers in your head, add them on an abacus and then try to add them again in your head. The result is 15. To record this result, move one line below where you recorded the result for the sum of the tens-column and record the result for the ones-column — a 5 in the ones-column and a 1 in the tens-column. To record the 1 in the tens column, go back up one row to the 3 and add a little +1 next to it. Now add $3 + 1$ and write the result in the tens-column in the same row where you just recorded the 5 in the ones-column. Draw a horizontal line under this row. To report the answer, go from highest column to the lowest while *looking up the column* for the first number you see. Record these numbers below the horizontal line. The resulting number is your answer, 45.

	•
1	3
1	6
1	3
	3
3^{+1}	
4	5
4	5

$$\boxed{68 + 94 + 7 + 39 = 208} \tag{9.5}$$

Line up the numbers in their respective columns. Since the tens-column is the highest column with any numbers in it, add these numbers first. The result is 18. Draw a horizontal line and write a 1 in the hundreds-column and a 8 in the tens-column. Now add the numbers in the ones-column. If you have difficulty adding these numbers in your head, add them on an abacus and then try to add them again in your head. The result is 28. To record this result, move one line below where you recorded the result for the sum of the tens-column and record the result for the ones-column — an 8 in the ones-column and a 2 in the tens-column. To record the 2 in the tens column, go back up one row to the 8 and add a little +2 next to it. Now add $8 + 2$ and write the result in the tens-column in the same row where you just recorded the 8 in the ones-column. To write this result, write a 0 in the tens-column, go back up one row to the 1 in the hundreds-column and add a little +1 next to it. Now add $1 + 1$ and write the result in the hundreds-column in the same row where you just recorded the 8 in the ones-column. Draw a horizontal line under this row. To report the answer, go from highest column to the lowest while *looking up the column* for the first number you see. Record these numbers below the horizontal line. The resulting number is your answer, 208.

		•
	6	8
	9	4
		7
	3	9
1^{+1}	8^{+2}	
2	0	8
2	0	8

$$\boxed{15.3 + 71.1 + 26.2 + 57.4 = 170.0} \tag{9.6}$$

Line up the numbers in their respective columns. Since the tens-column is the highest column with any numbers in it, add these numbers first. The result is 15. Draw a horizontal line and write a 1 in the hundreds-column and a 5 in the tens-column. Now add the numbers in the ones-column. If you have difficulty adding these numbers in your head, add them on an abacus and then try to add them again in your head. The result is 19. To record this result, move one line below where you recorded the result for the sum of the tens-column and record the result for the ones-column — a 9 in the ones-column and a 1 in the tens-column. To record the 1 in the tens-column, go back up one row to the 5 and add a little +1 next to it. Now add $5 + 1$ and write the result in the tens-column in the same row where you just recorded the 9 in the ones-column. Now add the numbers in the tenths-column. The result is 10. To record this result, move one line below where you recorded the result for the sum of the ones-column and record the result for the tenths-column — a 0 in the tenths-column and a 1 in the ones-column. To record the 1 in the ones-column, go back up one row to the 6 and add a little +1 next to it. Now add $6 + 1$ and write the result in the ones-column in the same row where you just recorded the 0 in the tenths-column. Draw a horizontal line under this row. To report the answer, go from highest column to the lowest while *looking up the column* for the first number you see. Record these numbers below the horizontal line. The resulting number is your answer, 170.0.

		•	
	1	5	3
	7	1	1
	2	6	2
	5	7	4
1	5^{+1}		
	6^{+1}	9^{+1}	
	7	0	0
1	7	0	0

$$\boxed{-43.93 + 64.77 + 80.46 - 18.95 = 82.35} \tag{9.7}$$

Line up the numbers in their respective columns. Since the tens-column is the highest column with any numbers in it, add these numbers first. The result is 15. Draw a horizontal line and write a 1 in the hundreds-column and a 5 in the tens-column. Now add the numbers in the ones-column. If you have difficulty adding these numbers in your head, add them on an abacus and then try to add them again in your head. The result is 19. To record this result, move one line below where you recorded the result for the sum of the tens-column and record the result for the ones-column — a 9 in the ones-column and a 1 in the tens-column. To record the 1 in the tens-column, go back up one row to the 5 and add a little +1 next to it. Now add $5+1$ and write the result in the tens-column in the same row where you just recorded the 9 in the ones-column. Now add the numbers in the tenths-column. The result is 10. To record this result, move one line below where you recorded the result for the sum of the ones-column and record the result for the tenths-column — a 0 in the tenths-column and a 1 in the ones-column. To record the 1 in the ones-column, go back up one row to the 6 and add a little +1 next to it. Now add $6+1$ and write the result in the ones-column in the same row where you just recorded the 0 in the tenths-column. Draw a horizontal line under this row. To report the answer, go from highest column to the lowest while *looking up the column* for the first number you see. Record these numbers below the horizontal line. The resulting number is your answer, 170.0.

	•		
-4	3	9	3
6	4	7	7
8	0	4	6
-1	8	9	5
9^{-1}			
8	3^{-1}		
	2	3	
			5
8	2	3	5

9.2 Multiplication

Multiplying on paper with abacus-thinking is different from multiplying on a real abacus in one important way: on the abacus you multiply from lowest to highest digit of the multiplicand, but on paper you multiply the highest to the lowest digit of the multiplicand.[68]

The trick to multiplying on paper is to know the multiplication tables up to 9×9 and to be very careful about which rods' numbers you are multiplying together at any time.

$$\boxed{3 \times 2 = 6} \tag{9.8}$$

Let's begin with a trivial example: $3 \times 2 = 6$. The multiplier is 3 and the multiplicand is 2. The highest rod in the multiplier is the ones-rod. The highest rod in the multiplicand is also the ones-rod. Multiply the rod-values together to determine where to place the result of multipliying 3 and 2: $1 \times 1 = 1$. On paper, draw columns to denote abacus rods. Mark the ones-rod with a dot. So, place the result of 3×2 on the ones-rod. The answer is 6.

•
6

$$\boxed{8 \times 7 = 56} \tag{9.9}$$

The next example is $8 \times 7 = 56$. The multiplier is 8 and the multiplicand is 7. The highest rod in the multiplier is the ones-rod. The highest rod in the multiplicand is also the ones-rod. Multiply the rod-values together to determine where to place the result of multipliying 8 and 7: $1 \times 1 = 1$. On paper, draw columns to denote abacus rods. Mark the ones-rod with a dot. So, place the result of 8×7 on the ones-rod. The answer is 56.

	•
5	6

$$\boxed{6 \times 39 = 234} \tag{9.10}$$

[68] Multiplying the highest digit first on the abacus would disturb the lower digits already set on the abacus. We can avoid this problem on paper, so we start multiplying the highest digit first.

The next example is $6 \times 39 = 234$. The multiplier is 6 and the multiplicand is 39. The highest rod in the multiplier is the ones-rod. The highest rod in the multiplicand is the tens-rod. Multiply the rod-values together to determine where to place the result of multipliying 6 and 3: $1 \times 10 = 10$. So, put 18 on the tens-rod. Now multiply 6×9. Since both numbers are on the ones-rod, the result must be put on the ones-rod. So, put 54 on the ones-rod. You can't read the answer yet; you must now add the numbers in the columns just as you learned in Lesson 9.1. After you do that, read the answer, 234.

		•
1	8	
	5	4
1^{+1}		
2	3	
		4
2	3	4

9.3 Division

9.4 Exercises

10 Mental Math

Have you calculated so much on your abacus that you can see the beads in your head? If so, try adding and subtracting on the abacus with your eyes closed. Now put your abacus away, close your eyes, and calculate on the abacus *in your mind*. As you calculate, move the fingers of your right hand as if you were moving the beads on your mental abacus — finger motion reinforces what you see on the abacus in your mind.

10.1 Exercises

Exercise 10.1

Here's a simple way to practice adding or subtracting single digits in your head. (This skill is indispensible for adding or subtracting on paper.) When you're on the road, add all the digits on a license plate or billboard you happen to see. If possible, challenge someone in your vehicle to do the same. You may compare answers or see who can calculate the answer first. For example, suppose the vehicle in front of you has the license plate $2Q4821$. Add $2+4+8+2+1$ in your head. You may choose to mix in negative numbers. Suppose you see a billboard with the phone number 289–2937. You could choose the last four digits to be negative numbers and calculate $2+8+9-2-9-3-7$ in your head. For this example, the answer is negative and you will also be required to decode the negative answer in your head.

Index